MATH348-Advanced Engineering Mathematics		**Homework: PDEs - Part II**
	CONSERVATION LAWS, SOURCE TERMS, DIFFUSION IN $\mathbb{R}^{2+1}$, CHAIN RULE, SL-PROBLEMS	
Text: 12.8,12.9	Lecture Notes : N/A	Lecture Slides: N/A

Quote of Homework Five
And the feeling is that there's something wrong, 'cause I can't find the words, and I can't find the songs.
Radiohead : Stop Whispering (1993)

1. Conservation Laws in One-Dimension

Recall that the conservation law encountered during the derivation of the heat equation was given by,

$$\frac{\partial u}{\partial t} = -\kappa \nabla \cdot \boldsymbol{\phi} = -\kappa \operatorname{div}(\boldsymbol{\phi}), \tag{1}$$

which reduces to

$$\frac{\partial u}{\partial t} = -\kappa \frac{\partial \phi}{\partial x}, \ \kappa \in \mathbb{R} \tag{2}$$

in one-dimension of space.[1] In general, if the function $u = u(x,t)$ represents the density of a physical quantity then the function $\phi = \phi(x,t)$ represents its flux. If we assume the ϕ is proportional to the negative gradient of u then, from (2), we get the one-dimensional heat/diffusion equation. [2]

1.1. **Transport Equation.** Assume that ϕ is proportional to u to derive, from (2), the convection/transport equation, $u_t + cu_x = 0 \ \ c \in \mathbb{R}$.

1.2. **General Solution to the Transport Equation.** Show that $u(x,t) = f(x - ct)$ is a solution to this PDE.

1.3. **Diffusion-Transport Equation.** If both diffusion and convection are present in the physical system then the flux is given by, $\phi(x,t) = cu - du_x$, where $c, d \in \mathbb{R}^+$. Derive from, (2), the convection-diffusion equation $u_t + \alpha u_x - \beta u_{xx} = 0$ for some $\alpha, \beta \in \mathbb{R}$.

1.4. **Convection-Diffusion-Decay.** If there is also energy/particle loss proportional to the amount present then we introduce to the convection-diffusion equation the term λu to get the convection-diffusion-decay equation.[3]

1.5. **General Importance of Heat/Diffusion Problems.** Given that,

$$u_t = Du_{xx} - cu_x - \lambda u. \tag{3}$$

Show that by assuming, $u(x,t) = w(x,t)e^{\alpha x - \beta t}$, (3) can be transformed into a heat equation on the new variable w where $\alpha = c/(2D)$ and $\beta = \lambda + c^2/(4D)$.[4]

2. One Dimensional Heat Equation with Source Term

Given,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} + F(x,t), \tag{4}$$

where $x \in (0, L)$ and $t \in (0, \infty)$, subject to

$$u_x(0,t) = 0, \ u_x(L,t) = 0, \tag{5}$$

and

$$u(x,0) = g(x). \tag{6}$$

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[1] When discussing heat transfer this is known as Fourier's Law of Cooling. In problems of steady-state linear diffusion this would be called Fick's First Law. In discussing electricity u could be charge density and q would be its flux.

[2] AKA Fick's Second Law associated with linear non-steady-state diffusion.

[3] The u_{xx} term models diffusion of energy/particles while u_x models convection, u models energy/particle loss/decay. The final term should not be surprising! Wasn't the appropriate model for radioactive/exponential decay $Y' = -\alpha^2 Y$?

[4] This shows that the general PDE (3), which models can be solved using heat equation techniques.

2.1. **Cosine Half-Range Expansion.** Let $F(x,t) = e^{-t}\sin\left(\frac{2\pi}{L}x\right)$ be the heat generation function. Find the Fourier cosine half-range expansion of F.

2.2. **General Solution.** Using the previous result, solve for $G_n(t)$ for $n = 0, 1, 2, 3, \ldots$ assuming that $u(x,t) = G_0(t) + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{L}x\right) G_n(t)$.

2.3. **Fourier Coefficients.** Assuming that $g(x) = \begin{cases} \frac{2k}{L}x, & 0 < x < \frac{L}{2}, \\ \frac{2k}{L}(L-x), & \frac{L}{2} < x < L \end{cases}$, solve for any unknown constants associated with the general solution.

3. Time Dependent Boundary Conditions

It makes sense to consider time-dependent interface conditions. That is, (4) and (6) subject to

$$u(0,t) = g(t), \;\; u(L,t) = h(t), \;\; t \in (0,\infty) \tag{7}$$

Show that this PDE transforms into:

$$\frac{\partial w}{\partial t} = c^2 \frac{\partial^2 w}{\partial x^2} - S_t(x,t) \quad , \tag{8}$$

$$x \in (0,L), \qquad t \in (0,\infty), \qquad c^2 = \frac{\kappa}{\rho\sigma}. \tag{9}$$

with boundary conditions and initial conditions,

$$w(0,t) = w(L,t) = 0, \tag{10}$$

$$w(x,0) = F(x), \tag{11}$$

where $F(x) = f(x) - S(x,0)$ and $S(x,t) = \frac{h(t)+g(t)}{L}x + g(t)$. [5]

4. Coordinate Systems, Multivariate Chain Rule and the Laplacian

Recall that the Laplacian, $\triangle u = u_{xx} + u_{yy} + u_{zz}$, was a general term in the heat equation in $\mathbb{R}^{3+1}$. This is especially nice in Cartesian coordinates but if you change coordinates then the multivariate chain rule must be used to convert the associated derivatives. For example in polar coordinates $r = \sqrt{x^2+y^2}$ and $u_r(x,y) = u_r r_x + u_r r_y$. For this reason the Laplacian changes form in cylindrical and spherical coordinates.

4.1. **Laplacian in Cylindrical Coordinates.** Show that if $x = r\cos(\theta)$ and $y = r\sin(\theta)$ then $\triangle u = u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} + u_{zz}$.

4.2. **Laplacian in Spherical Coordinates.** Show that if $x = \rho\cos(\theta)\sin(\phi)$, $y = \rho\sin(\theta)\sin(\phi)$ and $z = \rho\cos(\phi)$ then $\triangle u = u_{rr} + 2r^{-1}u_r + r^{-2}u_{\phi\phi} + r^{-2}\cot(\phi)u_\phi + r^{-2}\csc^2(\phi)u_{\theta\theta}$

5. Sturm-Liouville Problems

A Sturm-Liouville eigenproblem is given by,

$$Lu = \frac{1}{w(x)}\left(-\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u\right) = \lambda u, \;\; \lambda \in \mathbb{C} \tag{12}$$

whose nontrivial eigenfunctions must satisfy the boundary conditions,

$$l_1 u(a) + l_2 u'(a) = 0 \tag{13}$$

$$r_1 u(b) + r_2 u'(b) = 0. \tag{14}$$

5.1. **Orthogonality of Solutions: Special Case.** Let $l_2 = r_2 = 0$, $a = 0$, $b = \pi$, $w(x) = 1$, $p(x) = 1$ and $q(x) = 0$ and show that (12) with (13)-(14) defines a set of an orthogonal functions.

5.2. **Orthogonality of Solutions: General Case (Extra Credit).** Let (λ_1, u_1) and (λ_2, u_2) be two different eigenvalue/eigenfunction pairs. Show that u_1 and u_2 are orthogonal. That is, show that $\langle u_1, u_2\rangle = 0$ with respect to the inner-product defined by $\langle f, g\rangle = \int_a^b f(x)g(x)dx$.

5.3. **Bessel's Equation.** Show that if $p(x) = x$, $q(x) = \nu^2/x$ and $w(x) = x/\lambda$ then (12) becomes $x^2u'' + xu' + (x^2 - \nu^2)u = 0$, which is known as Bessel's equation of order ν.

[5] A similar transformation can be found for the wave equation with inhomogeneous boundary conditions. The moral here is that time-dependent boundary conditions can be transformed into externally driven (AKA Forced or inhomogeneous) PDE with standard boundary conditions.

5.4. **Fourier Bessel Series.** A solution to Bessel's equation is for $\nu = n \in \mathbb{N}$,

(15) $$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!}, \quad n = 1, 2, 3, \ldots$$

which is called Bessel's function of the first-kind of order n. Since these functions manifest from a SL problem they naturally orthogonal and have an orthogonality condition,

(16) $$\langle J_n(xk_{n,m}), J_n(xk_{n,i})\rangle = \int_0^R xJ_n(xk_{n,m})J_n(xk_{n,i})dx = \frac{\delta_{mi}}{2}\left[RJ_{n+1}(k_{nm}R)\right]^2.$$

Using this show that the coefficients in the Fourier-Bessel series,

(17) $$f(x) = \sum_{m=1}^{\infty} a_m J_n(k_{n,m}x),$$

are given by,

(18) $$a_i = \frac{2}{R^2 J_{n+1}^2(k_{n,m}R)} \int_0^R xJ_n(k_{n,i}R)f(x)dx, \quad i = 1, 2, 3, \ldots$$