Again this equation can be integrated yielding

$$\frac{dx_n(t+T)}{dt} = c \ln |x_n(t) - x_{n-1}(t)| + d_n.$$

Let us consider a steady-state situation, in which case

$$u = -c \ln \rho + d.$$

Once more the integration constant is chosen such that at maximum density, the velocity is zero. In that way

$$u = -c \ln \frac{\rho}{\rho_{max}},$$

sketched in Fig. 64-3. Difficulties as $\rho \longrightarrow 0$ are again avoided by assuming

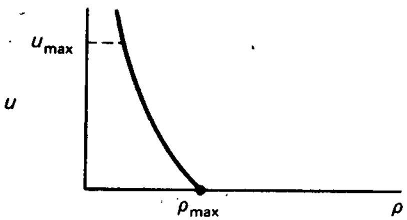

Figure 64-3 Nonlinear steady-state car-following model: velocity-density relationship.

that for low densities, $u = u_{max}$. The constant c is chosen (perhaps by a least-squares fit) so that the formula agrees well with observed data on a given highway. This formula can be shown to agree quite well with the observed data. The constant c has a simple interpretation, namely we will now show it is the velocity corresponding to the maximum flow:

$$q = \rho u = -c\rho \ln \frac{\rho}{\rho_{max}} \quad \text{and} \quad 0 = \frac{dq}{d\rho} = -c\left(\ln \frac{\rho}{\rho_{max}} + 1\right)$$

imply that the maximum traffic flow occurs at $\rho = \rho_{max}/e$, in which case the velocity at the maximum flow is c,

$$u\left(\frac{\rho_{max}}{e}\right) = c.$$

Many other types of similar car-following theories have been formulated by traffic researchers. They help to explain the relationship between the individual action of single drivers and their collective behavior described by the velocity-density curve.

EXERCISES

64.1. Consider the linear car-following model, equation 64.2, with a response time T (a delay).

(a) Solve for the velocity of the nth car, $v_n = dx_n/dt$. Show that

$$\frac{dv_n(t+T)}{dt} = -\lambda(v_n(t) - v_{n-1}(t)).$$

(b) Assume the lead driver's velocity varies periodically

$$v_0 = Re\,(1 + e^{i\omega t}).$$

Also assume the nth driver's velocity varies periodically

$$v_n = Re\,(1 + f_n e^{i\omega t}),$$

where f_n measures the amplification or decay which occurs. Show that

$$f_n = \left(1 + \frac{i\omega}{\lambda} e^{i\omega T}\right)^{-n} f_0,$$

where $f_0 = 1$.

(c) Show the magnitude of the amplification factor f_n decreases with n if

$$\frac{\sin \omega T}{\omega} < \frac{1}{2\lambda}.$$

(d) Show that the above inequality holds for all ω only if $\lambda T < \frac{1}{2}$.

(e) Conclude that if the product of the sensitivity and the time lag is greater than $\frac{1}{2}$, it is possible for following cars to drive much more erratically than the leader. In this case we say the model predicts *instability* if $\lambda T > \frac{1}{2}$ (i.e., with a sufficiently long time lag). (This conclusion can be reached more expeditiously through the use of Laplace transforms.)

64.2. Consider the linear car-following model, equation 64.2, with a response time T (a delay).

(a) Solve for the velocity of the nth car, $v_n = dx_n/dt$. Thus show that

$$\frac{dv_n(t+T)}{dt} = -\lambda(v_n(t) - v_{n-1}(t)).$$

(b) Consider two cars only, the leader and the follower. Thus

$$\frac{dv_1(t+T)}{dt} + \lambda v_1(t) = \lambda v_0(t).$$

Look for homogeneous solutions ($v_0(t) = 0$) of the form e^{rt}. Show that these solutions are exponentially damped if $1/e > \lambda T > 0$.

(c) For what values of λT do solutions exist of the form e^{rt} with r complex, such that the solutions are oscillatory with growing, decaying, or constant amplitude?

As with exercise 64.1, the use of Laplace transforms simplifies the above calculations.

64.3. General car-following models of the following form can be considered:

$$\frac{d^2x_{n+1}(t+T)}{dt^2} = a\left(\frac{dx_{n+1}(t)}{dt}\right)^m \frac{\frac{dx_n}{dt}(t) - \frac{dx_{n+1}}{dt}(t)}{[x_n(t) - x_{n+1}(t)]^l}.$$

Note that the linear model, equation 64.2, corresponds to $m = 0$, $l = 0$ and the inverse-spacing model, equation 64.5, $m = 0$, $l = 1$.