5. Give $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

a) $\det(A - \lambda I) = \det\left(\begin{bmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{bmatrix}\right) =$

$= (4-\lambda)\{(1-\lambda)(1-\lambda)\} + 1\cdot\{-2(1-\lambda)\} = 0 \Leftrightarrow$

$\Leftrightarrow 4-\lambda\{(1-\lambda)^2\} = -2(1-\lambda)$, $\lambda = 1 \Rightarrow 0=0$

$\Leftrightarrow (4-\lambda)(1-\lambda) = -2 \Leftrightarrow$

$\Leftrightarrow 4+\lambda^2 - 5\lambda + 2 = 0 \Leftrightarrow$

$\Leftrightarrow \lambda^2 - 5\lambda + 6 = 0 \Rightarrow \lambda = 2, 3$

Thus, A has three eigenvalues.

$\lambda_1 = 1$

$\lambda_2 = 2$

$\lambda_3 = 3$

b)

<u>Case $\lambda = 1$</u>:

$[A - \lambda I \mid 0] = \left[\begin{array}{ccc|c} 3 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 3 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$$\Rightarrow \begin{array}{l} -2x_1 = 0 \\ 3x_1 = -x_3 \\ x_2 = \text{free} \end{array} \Rightarrow \vec{x} = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The basis for the Eigenspace associated with $\lambda = 1$ is,

$$B_{\lambda=1} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Case $\lambda = 2$:

$$[A - \lambda I \mid 0] = \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 \\ -2 & 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = -\frac{1}{2}x_3 \\ x_2 = x_3 \\ x_3 \text{ is free} \end{array} \quad \vec{x} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix} x_3$$

The basis for this case is $B_{\lambda=2} = \left\{ \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \right\}$

Case $\lambda = 3$:

$$[A - \lambda I \mid 0] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -2 & -2 & 0 & 0 \\ -2 & 0 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$$

$$\Rightarrow \quad \begin{aligned} x_1 &= -x_3 \\ x_2 &= x_3 \\ x_3 &\text{ is free} \end{aligned} \quad \Rightarrow \quad \vec{x} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} x_3$$

$\Rightarrow$ Basis for Eigenspace of A when $\lambda = 3$ is

$$B_{\lambda=3} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

3. Yes, A is diagonalizable. Why? The following process will produce 4-linearly independent Eigenvectors

1^{st}: Since A is triangular we know the Eigenvalues of A are

$\lambda_1 = 4$ (with algebraic multiplicity 2)
$\lambda_2 = 2$ (with algebraic multiplicity 2)

2^{nd}:

Case $\lambda = 4$

$$[A-4I \mid 0] = \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 \end{array}\right] \Rightarrow$$

$\Rightarrow$ $x_3 = 0$ $\Rightarrow$ $\vec{x} = \begin{bmatrix} 2x_4 \\ x_2 \\ 0 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$
$x_1 = 2x_4$
x_2, x_4 free

Thus the Basis for the Eigenspace is, $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} = B_{\lambda=4}$

Case $\lambda = 2$

$$[A-4I \mid 0] = \left[\begin{array}{cccc|c} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3 = \text{free} \\ x_4 = \text{free} \end{array} \Rightarrow \vec{x} = \begin{bmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

· Basis for this Eigenspace is , $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} = B_{\lambda=2}$

This implies that

$$P = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

1. Given that $A_{m\times n}$.

a) $\dim(\text{Row}A)$ = # of nonzero Rows in echelon form of A =
= number of pivot columns of A =

$\dim(\text{Nul}A)$ = # of zero Rows in an echelon form of A =
= number of nonpivot columns.

$\Rightarrow$ $\dim(\text{Row}A) + \dim(\text{Nul}A)$ = number of columns of $A = n$.

b)

$\dim(\text{Col}\,A)$ = number of pivot columns =
= number of nonzero Rows of an echelon form of A.

$\dim(\text{Nul}\,A^T)$ = number of nonpivot columns of echelon form of A =
= number of zero Rows of A

$\Rightarrow$ $\dim(\text{Col}A) + \dim(\text{Nul}A^T)$ = number of Rows of A.

Thus works if $m=n$ i.e. A is Square

($\Rightarrow$)
c. Assume $A\vec{x}=\vec{b}$ has a solution for all $\vec{b}\in\mathbb{R}^m$ then A is invertible, which implies that A^T is invertible. If A^T is invertible then the only soln to $A^Tx=0$ is the trivial one.

($\Leftarrow$)
Assume $A^T\vec{x}=0$ has only the trivial soln. This implies that A^T is invertible. If $(A^T)^{-1}$ Exists then A^{-1} Exists. If A^{-1} Exists $A\vec{x}=\vec{b}$ has a solution for all $b\in\mathbb{R}^n$.

$\square$

Case $m \neq n$.

($\Rightarrow$) Since $A\vec{x} = \vec{b}$ has a solution for each $\vec{b} \in \mathbb{R}^m$, there is a pivot position in each row of A. Thus, $\dim(\text{Row } A) = m$

By part a $\dim(\text{Nul} A^T) = m - \dim(\text{Col} A) = m - \dim(\text{Row} A) =$

$= m - m = 0$

Hence $\dim(\text{Nul} A^T) = 0 \Rightarrow \text{Nul} A^T = \{\vec{0}\}$.

($\Leftarrow$) $\dim(\text{Nul} A^T) = 0 \Rightarrow \dim(\text{Row} A) + 0 = m \Rightarrow$

$\Rightarrow$ pivot position in every row of $A \Rightarrow A\vec{x} = \vec{b}$ has a soln for each $\vec{b} \in \mathbb{R}^m$.

3d) $\det(A-\lambda I) = \det((A-\lambda I)^T) = \det(A^T-(\lambda I)^T) =$

$= \det(A^T - \lambda I^T) = \det(A^T - \lambda I)$

$\Rightarrow$ A has the same Char. poly as A^T

e) Given $A_{n \times n}$ assume A^{-1} exists Then,

$A\vec{x} = \lambda\vec{x} \iff A^{-1}A\vec{x} = I\vec{x} = \vec{x} = A^{-1}(\lambda\vec{x}) = \lambda A^{-1}\vec{x}$

$\iff \quad A^{-1}\vec{x} = \frac{1}{\lambda}\vec{x}$, provided $\lambda \neq 0$.

However since A^{-1} exists

$\Rightarrow \det(A) \neq 0 \Rightarrow \lambda \neq 0$.

4. a. Assume that there Exists P, D such that

$$A = PDP^{-1}$$

where P is invertible and D is diagonal. Then since A is invertible

$$A^{-1} = (PDP^{-1})^{-1} = (P^{-1})^{-1} D^{-1} P^{-1} = PD^{-1}P^{-1} \quad (1)$$

Since A^{-1} Exist $\lambda = 0$ is not an Eigenvalue of A and D has no zero Entries on it's diagonal. Since D has no zero Entries on it's diagonal D^{-1} Exists and is diagonal. Thus A^{-1} is also diagonalizable since it has the form PDP^{-1}!

b. If A has n-linearly independent Eigenvectors A is diagonalizable and there Exists P, D such that $A = PDP^{-1}$.

Then since P is invertible

$$A^T = (PDP^{-1})^T = (P^{-1})^T D^T P^T = (P^T)^{-1} D^T P^T$$

Noting that $D^T = D$ since D is diagonal implies

$$A^T = Q D Q^{-1}, \text{ where } Q = (P^T)^{-1}$$

which shows A^T is diagonalizable and by theorem 5.3.5 has n-linearly independent Eigenvectors

2\. a. Yes. The matrix is a stochastic matrix b/c it's columns are probability vectors. It is Regular since P^1 has all nonnegative Entries.

b. $P\vec{q} = \vec{q} \iff (P - I)\vec{q} = 0 \iff$

$$\iff \begin{bmatrix} -.9 & .6 & | & 0 \\ .9 & -.6 & | & 0 \end{bmatrix} \sim \begin{bmatrix} -.9 & .6 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$\Rightarrow .9q_1 = .6q_2$ $\Rightarrow$ $\vec{q} = \begin{bmatrix} 2/3\, q_2 \\ q_2 \end{bmatrix} = q_2 \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$

q_2 is free

Choose q_2 to be $q_2 + \frac{2}{3} q_2 = 1 \iff q_2(5/3) = 1 \iff q_2 = \frac{3}{5}$

$q_2 = 3/5$

$\Rightarrow$

$$\vec{q} = \begin{bmatrix} 6/15 \\ 3/5 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix}$$

Note

$$P\vec{q} = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix} \checkmark$$

c) Step 1 - Diagonalize P

$$\det\left(\begin{bmatrix} .1-\lambda & .6 \\ .9 & .4-\lambda \end{bmatrix}\right) = (.4-\lambda)(.1-\lambda) - .54 = \lambda^2 - .5\lambda - .54 + .04$$

$$= \lambda^2 - .5\lambda - .5 \Rightarrow \lambda_1 = 1$$

$$\lambda = \frac{-(-.5) \pm \sqrt{(-.5)^2 - 4(1)(-.5)}}{2(1)} = \frac{.5 \pm 1.5}{2} = 1, -.5$$

$\lambda_1 = 1$

$\lambda_2 = -.5$

Case $\lambda = 1$, $(P - I)\vec{x} = 0 \Rightarrow x = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix}$

Case $\lambda = -.5$

$(P - -.5I)\vec{x} = 0$

$\Rightarrow$

$$\left[\begin{array}{cc|c} .6 & .6 & 0 \\ .9 & .9 & 0 \end{array}\right] \Rightarrow \vec{x} = x_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Thus

$$P = \begin{bmatrix} 2/5 & 1 \\ 3/5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -.5 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -3/5 & 2/5 \end{bmatrix} \cdot \frac{1}{-1}$$

and

$$P^k = \begin{bmatrix} 2/5 & 1 \\ 3/5 & -1 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & -.5^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2/5 \end{bmatrix}$$

Thus,

$$\lim_{k \to \infty} P^k = \begin{bmatrix} 2/5 & 1 \\ 3/5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2/5 \end{bmatrix} = \begin{bmatrix} .4 & .4 \\ .6 & .6 \end{bmatrix}$$

This implies that for $x_0 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $x_1 + x_2 = 1$

$$\lim_{k\to\infty} x_k = \lim_{k\to\infty} B D^k B^{-1} x_0 = \begin{bmatrix} .4 & .4 \\ .6 & .6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} .4(x_1+x_2) \\ .6(x_1+x_2) \end{bmatrix} = \begin{bmatrix} .4 \\ .6 \end{bmatrix}$$

5. a. To find the Eigenfunctions we find the Eigenvalues and eigenvectors of A.

$$\det(A-\lambda I) = (3-\lambda)(1-\lambda) - (-2) = \lambda^2 - 4\lambda + 5$$

$$\lambda = \frac{-(-4) \pm \sqrt{16-4(1)(5)}}{2} = 2 \pm i$$

Case $\lambda = 2+i$:

$$[A-\lambda I \mid 0] = \left[\begin{array}{cc|c} 3-(2+i) & 1 & 0 \\ -2 & 1-(2+i) & 0 \end{array}\right] =$$

$$= \left[\begin{array}{cc|c} 1-i & 1 & 0 \\ -2 & -1-i & 0 \end{array}\right] \Rightarrow (1-i)x_1 + 1x_2 = 0$$ if $x \in Nul(A-\lambda I)$ then this relationship must hold.

Let $x_1 = -1$ $\Rightarrow x_2 = (1-i)$ $\Rightarrow \vec{x} = \begin{bmatrix} -1 \\ 1-i \end{bmatrix}$

Case $\lambda = 2-i$ $\Rightarrow \vec{x} = \begin{bmatrix} -1 \\ 1+i \end{bmatrix}$

Thus the Eigenfunctions of $\vec{x}' = A\vec{x}$ are given as,

$$\vec{x}_1(t) = \begin{bmatrix} -1 \\ 1-i \end{bmatrix} e^{(2+i)t}, \quad \vec{x}_2(t) = \begin{bmatrix} -1 \\ 1+i \end{bmatrix} e^{(2-i)t}$$

b. Using the formula pg 359 we have,

Real valued general soln

$$\vec{X}(t) = c_1\vec{Y}_1(t) + c_2\vec{Y}_2(t)$$

$$= c_1\left\{\begin{bmatrix}-1\\1\end{bmatrix}\cos t - \begin{bmatrix}0\\-1\end{bmatrix}\sin(t)\right\}e^{2t} +$$

$$+ c_2\left\{\begin{bmatrix}-1\\1\end{bmatrix}\cos t + \begin{bmatrix}0\\-1\end{bmatrix}\sin(t)\right\}e^{2t}$$