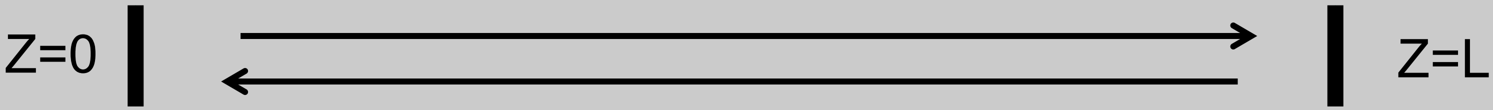
Example: linear resonator (1D)



- Boundary conditions: conducting ends (mirrors)

$$E_x\left(z=0,t\right)=0 \qquad E_x\left(z=L_z,t\right)=0$$

- Field is a superposition of +'ve and –'ve waves:

$$E_x\left(z,t\right)=A_+e^{i\left(k_z z-\omega t+\phi_+\right)}+A_-e^{i\left(-k_z z-\omega t+\phi_-\right)}$$

- Absorb phase into complex amplitude

$$E_x\left(z,t\right)=\left(A_+e^{+ik_z z}+A_-e^{-ik_z z}\right)e^{-i\omega t}$$

- Apply b.c. at z = 0

$$E_x\left(0,t\right)=0=\left(A_++A_-\right)e^{-i\omega t}\rightarrow A_+=-A_-$$

$$E_x\left(z,t\right)=A\sin k_z z\; e^{-i\omega t}$$

Quantization of frequency: longitudinal modes

- Apply b.c. at far end

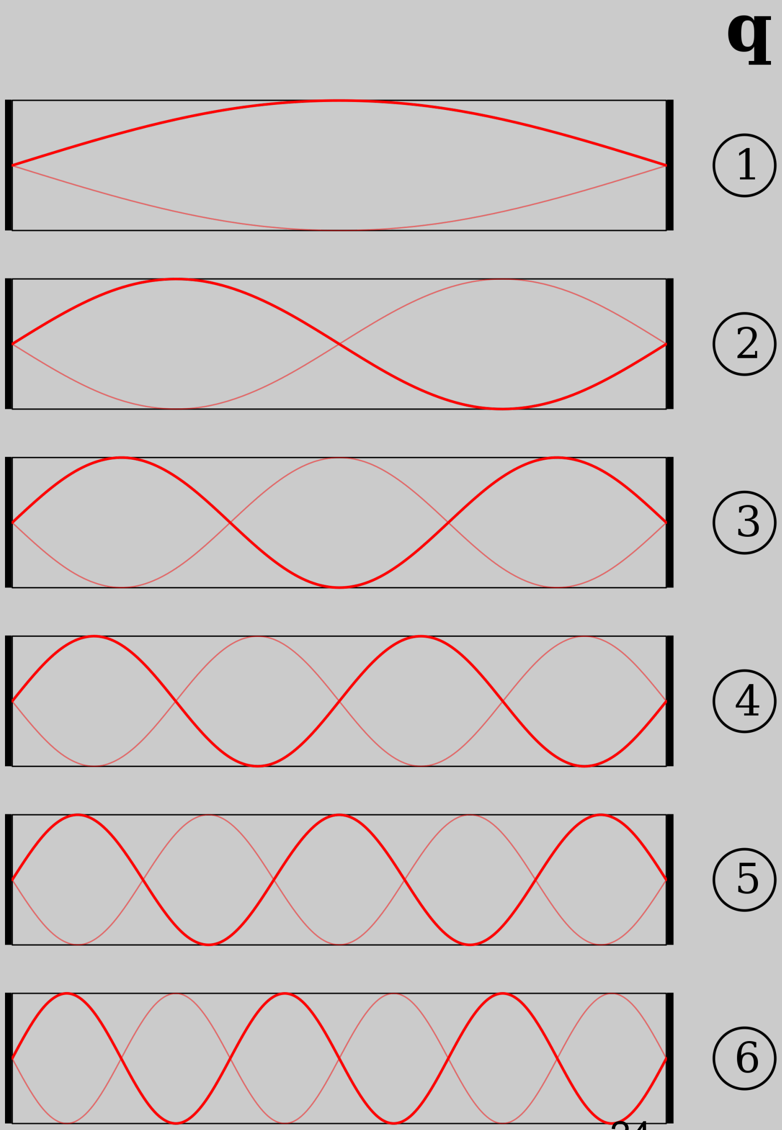
$$E_x\left(L_z,t\right)=0=A\sin k_z L_z\, e^{-i\omega t}$$

$$\rightarrow k_z L_z = q\pi \qquad q=1,2,3,\cdots$$

- Relate to wavelength:

$$k_z=\frac{2\pi}{\lambda}=\frac{q\pi}{L_z}\rightarrow L_z=q\frac{\lambda}{2}$$

Integer number of half-wavelengths fit in the resonator



Quantization of frequency: longitudinal modes

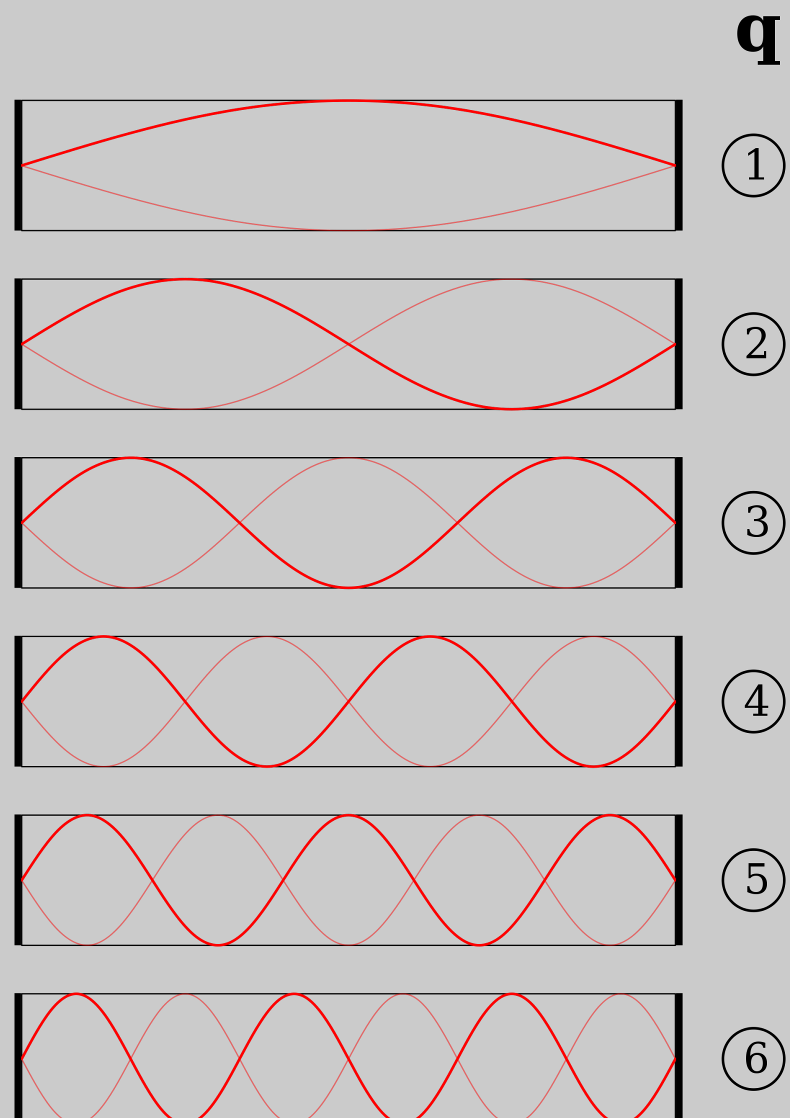
- Relate allowed wavelengths to frequency:

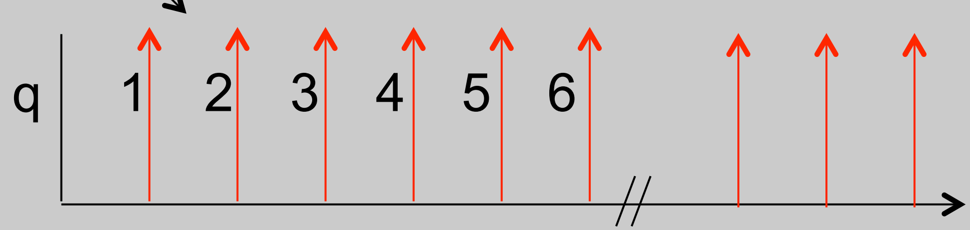
$$k_z = \frac{2\pi}{\lambda} = \frac{q\pi}{L_z} \rightarrow L_z = q\frac{\lambda}{2}$$

$$\frac{\omega_q}{c} = \frac{q\pi}{L_z} \rightarrow \nu_q = q\frac{c}{2L_z}$$

$$\Delta\nu = \frac{c}{2L_z} = \frac{1}{T_{RT}}$$

Frequency spacing = 1/ round trip time





2D resonator

- Assume separable function

$$\mathbf{E}(x,y,t) \sim f_1(x) f_2(y) g(t)$$

$$\vec{\nabla}^2 \mathbf{E}(x,y,t) = \frac{\partial^2}{\partial x^2}\mathbf{E}(x,y,t) + \frac{\partial^2}{\partial y^2}\mathbf{E}(x,y,t) = \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E}(x,y,t)$$

- Solution takes the form:

$$\mathbf{E}(x,y,t) = \mathbf{E_0} e^{ik_x x} e^{ik_y y} e^{-i\omega t} = \mathbf{E_0} e^{i\left(k_x x + k_y y\right)} e^{-i\omega t}$$

$$\mathbf{E}(x,y,t) = \mathbf{E_0} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

  – Now k-vector can point in arbitrary direction

- With this solution in W.E.:

$$\boxed{n^2 \frac{\omega^2}{c^2} = k_x^2 + k_y^2 = \mathbf{k}\cdot\mathbf{k}}$$

Valid even in waveguides and resonators

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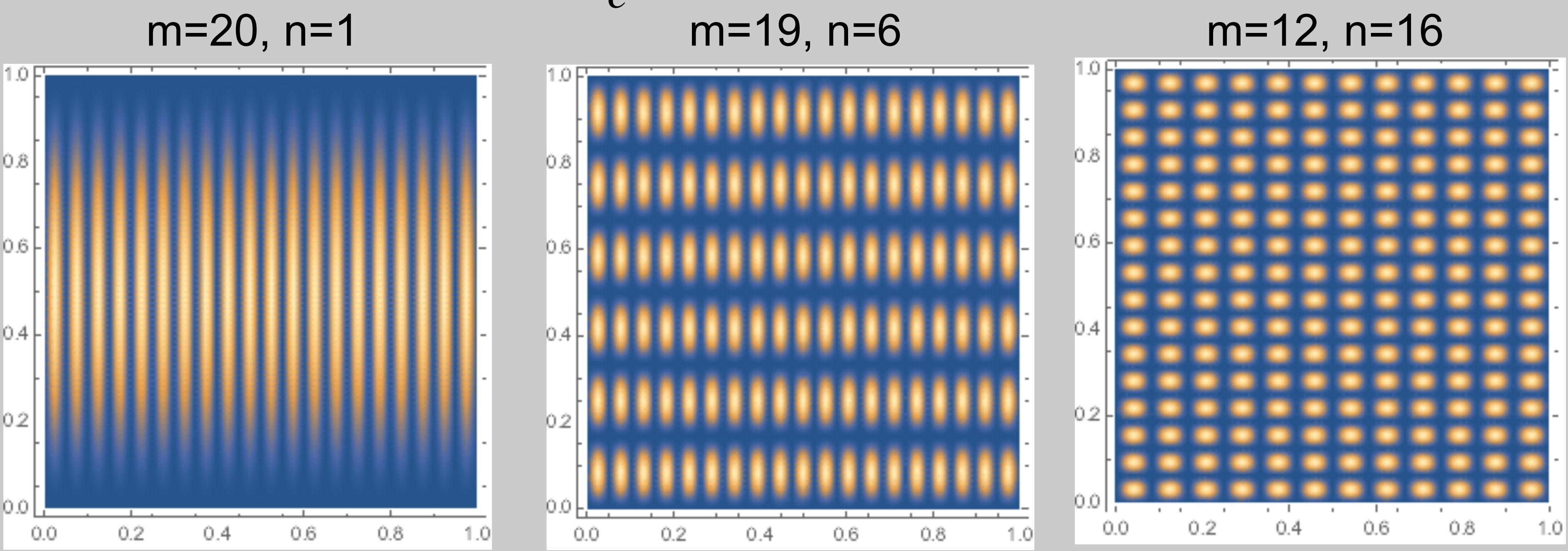
2D resonator: fixed frequency

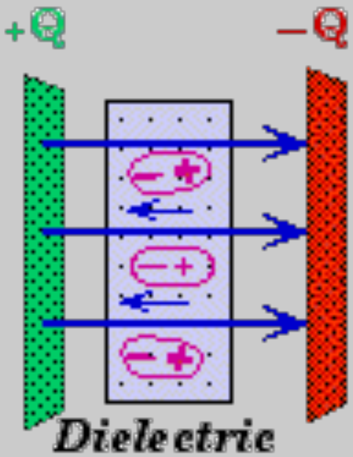
- If we fix the frequency, what modes can still fit?
- Example, n=1, c=1, L=1, k_x=m π, k_y=n π
  - If m=20, n=1, $$n^2 \frac{\omega^2}{c^2} = k_x^2 + k_y^2 \rightarrow \omega = \pi\sqrt{m^2+n^2} \approx 20\pi$$



All of these have close to the same frequency

Energy in EM waves and fields

- Statics: work is done to set up E or B fields
  - Charged capacitor: where does the energy get stored?
    - Vacuum, energy is in the electric field
    - Dielectric, energy also in polarized medium
  - Inductor: energy in field and magnetization
- Waves: energy stored in both E, B fields
  - Traveling wave: Poynting vector = vector form of intensity. Energy is transported by beam
  - Standing wave: stationary energy density pattern



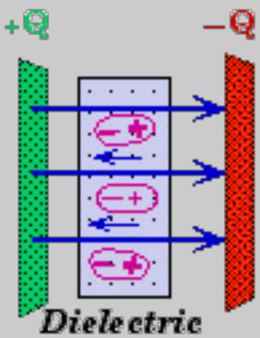
Wave energy and intensity

- Both E and H fields have a corresponding energy density (J/m^3)
  - For static fields (e.g. in capacitors) the energy density can be calculated through the work done to set up the field

$$\rho = \tfrac{1}{2}\varepsilon E^2 + \tfrac{1}{2}\mu H^2$$

  - Some work is required to polarize the medium
  - Energy is contained in both fields, but H field can be calculated from E field

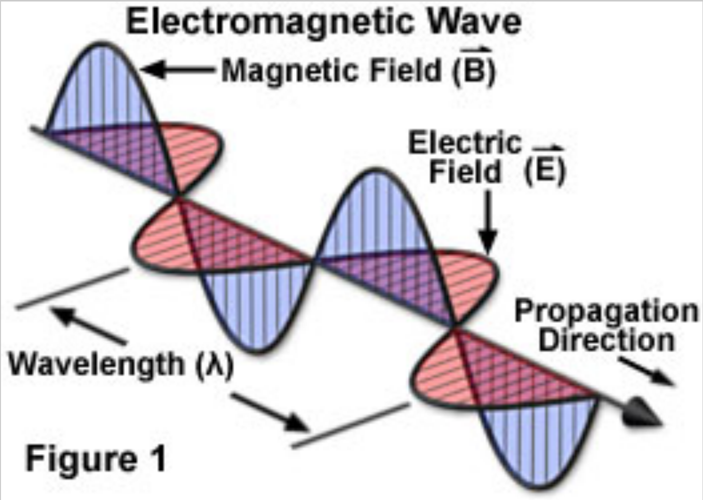
Work done charging a capacitor



H field from E field

- Maxwell equations relate E and H fields
- H field for a propagating wave is *in phase* with E-field $\mathbf{E} = \hat{\mathbf{x}} E_0 \exp\left[i\left(kz - \omega t \right) \right]$

$$\mathbf{H} = \hat{\mathbf{y}} H_0 \exp\left[i\left(kz - \omega t \right) \right]$$



$$-\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E}$$

$$i\omega \mu_0 \mathbf{H} = i\mathbf{k} \times \mathbf{E}$$

$$\mathbf{H} = \hat{\mathbf{y}} \frac{k}{\omega \mu_0} E_0 \exp\left[i\left(kz - \omega t \right) \right]$$

- Amplitudes are not independent

$$H_0 = \frac{n}{c\mu_0} E_0 = n\varepsilon_0 c E_0$$

Energy density in an EM wave

- The energy of the EM wave resides in both E and H fields
- Energy density (J/m³)

$$\rho = \tfrac{1}{2}\varepsilon E^2 + \tfrac{1}{2}\mu_0 H^2$$

$$H = n\varepsilon_0 cE$$

$$\varepsilon = \varepsilon_0 n^2$$

$$\rho = \tfrac{1}{2}\varepsilon_0 n^2 E^2 + \tfrac{1}{2}\mu_0 n^2 \varepsilon_0{}^2 c^2 E^2$$

$$\mu_0 \varepsilon_0 c^2 = 1$$

$$\rho = \varepsilon_0 n^2 E^2 = \varepsilon_0 n^2 E^2 \cos^2\left(k_z z - \omega t\right)$$

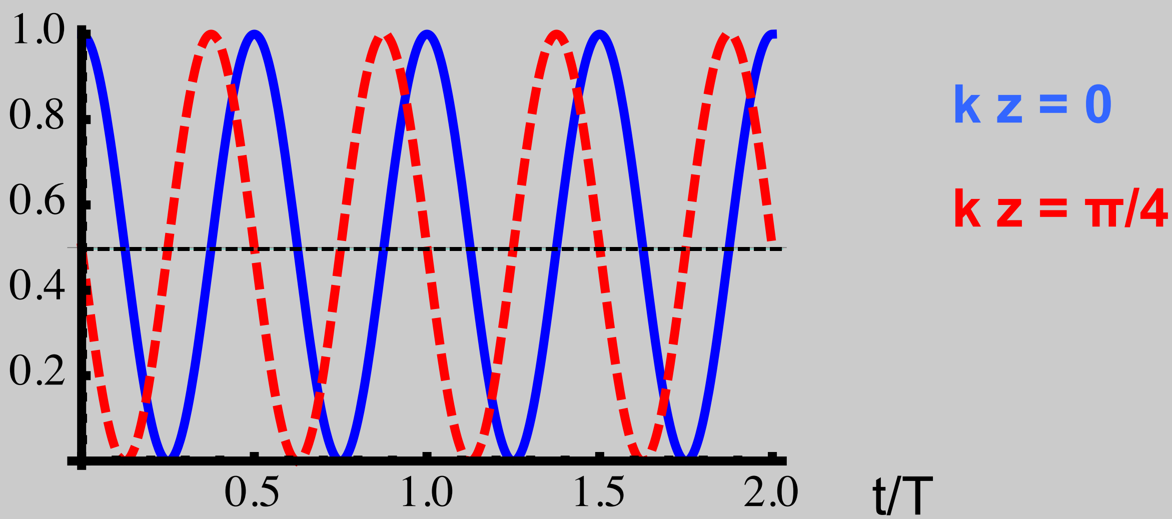
Equal energy in both components of wave

Cycle-averaged energy density

- Optical oscillations are faster than detectors
- Average over one cycle:

$$\langle \rho \rangle = \varepsilon_0 n^2 E_0{}^2 \frac{1}{T}\int_0^T \cos^2\left(k_z z - \omega t\right) dt$$

  – Graphically, we can see this should = ½

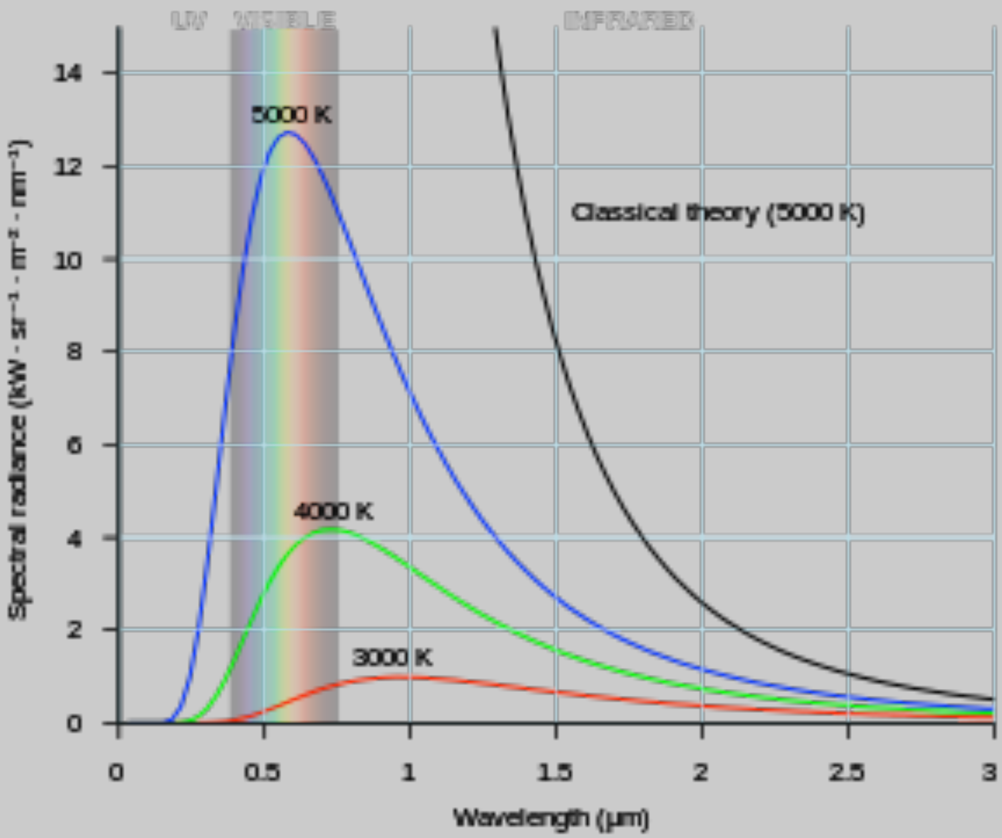


  – Independent of position z
  – Independent of frequency ω

$$\langle \rho \rangle = \frac{1}{2}\varepsilon_0 n^2 E_0{}^2$$

Blackbody radiation

- A perfect "blackbody" is in thermal equilibrium with its surroundings
  - Absorbs all incoming light
  - Smooth radiation curve



General 3D plane wave solution

- Assume separable function

$$\mathbf{E}(x,y,z,t) \sim f_1(x) f_2(y) f_3(z) g(t)$$

$$\vec{\nabla}^2 \mathbf{E}(\mathbf{r},t) = \frac{\partial^2}{\partial x^2}\mathbf{E}(\mathbf{r},t) + \frac{\partial^2}{\partial y^2}\mathbf{E}(\mathbf{r},t) + \frac{\partial^2}{\partial z^2}\mathbf{E}(\mathbf{r},t) = \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E}(z,t)$$

- Solution takes the form:

$$\mathbf{E}(x,y,z,t) = \mathbf{E_0} e^{ik_x x} e^{ik_y y} e^{ik_z z} e^{-i\omega t} = \mathbf{E_0} e^{i\left(k_x x + k_y y + k_z z\right)} e^{-i\omega t}$$

$$\mathbf{E}(x,y,z,t) = \mathbf{E_0} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

  – Now k-vector can point in arbitrary direction

- With this solution in W.E.:

$$\boxed{n^2 \frac{\omega^2}{c^2} = k_x^2 + k_y^2 + k_z^2 = \mathbf{k}\cdot\mathbf{k}}$$

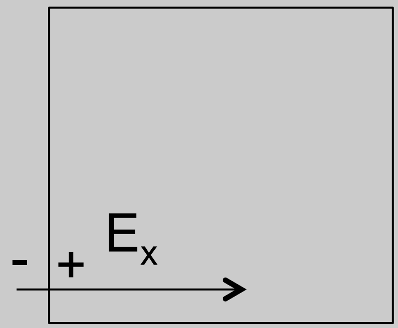
Valid even in waveguides and resonators

Closed box resonator: blackbody cavity

- Here we have a 3D pattern of standing waves
  - Exact boundary conditions aren't important, but for conducting walls:
    - E=0 where field is parallel to wall
    - Slope E=0 where field is perp to wall (charges can accumulate there)
  - Example standing wave solution:

$$E_x(x,y,z) = A_x \cos k_x x \, \sin k_y y \, \sin k_z z$$

    - Cos( ) function along field direction



  - Others:

$$E_y(x,y,z) = A_y \sin k_x x \cos k_y y \, \sin k_z z$$

$$E_z(x,y,z) = A_z \sin k_x x \, \sin k_y y \, \cos k_z z$$

Discrete wavevectors

- Discrete values of k:

$$k_x = \frac{l\pi}{L_x} \qquad k_y = \frac{m\pi}{L_y} \qquad k_z = \frac{n\pi}{L_z}$$

- With these solutions in the wave equation

$$\frac{\omega^2}{c^2} = k_x^2 + k_y^2 + k_z^2 = \mathbf{k} \cdot \mathbf{k}$$

2 allowed polarizations

  – k's are discrete, so there are discrete allowed frequencies:

$$\omega_{lmn} = c\sqrt{k_x^2 + k_y^2 + k_z^2} = c\sqrt{\left(\frac{l\pi}{L_x}\right)^2 + \left(\frac{m\pi}{L_y}\right)^2 + \left(\frac{n\pi}{L_z}\right)^2}$$

$$\nu_{lmn} = \frac{c}{2\pi}\sqrt{k_x^2 + k_y^2 + k_z^2} = c\sqrt{\left(\frac{l}{2L_x}\right)^2 + \left(\frac{m}{2L_y}\right)^2 + \left(\frac{n}{2L_z}\right)^2}$$

Field in equilibrium with walls: classical

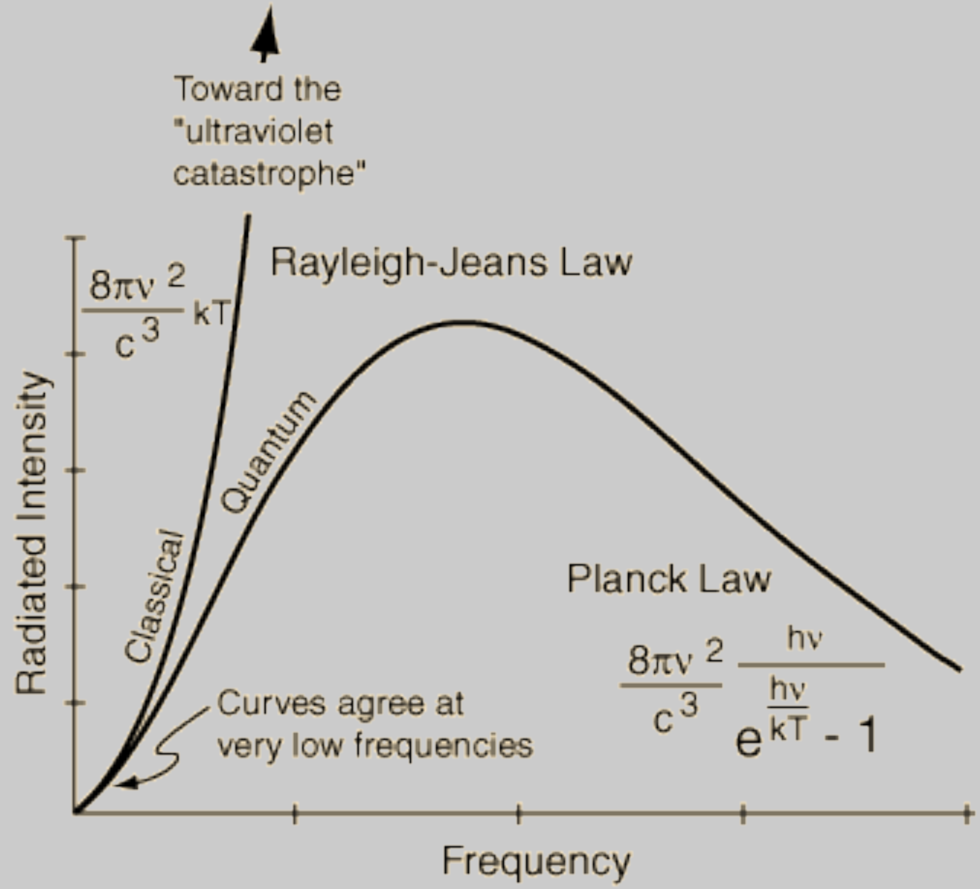
- Hold cavity walls at temperature T
- What is probability that a mode will be excited?
- Classical view (Boltzmann): $P(\mathcal{E}) \propto e^{-\mathcal{E}/k_BT}$
  - assume the amount of energy in each mode can take any value (continuous range) this is wrong!
  - average energy for *each* mode is

$$\langle \mathcal{E} \rangle = \frac{\int_0^\infty \mathcal{E}\, P(\mathcal{E})\, d\mathcal{E}}{\int_0^\infty P(\mathcal{E})\, d\mathcal{E}} = \frac{\int_0^\infty \mathcal{E}\, e^{-\mathcal{E}/k_BT}\, d\mathcal{E}}{\int_0^\infty e^{-\mathcal{E}/k_BT}\, d\mathcal{E}} = k_BT$$

  - Note: this is not $k_BT/2$ as in equipartition of K.E. There, integrate on velocity, which ranges – to +

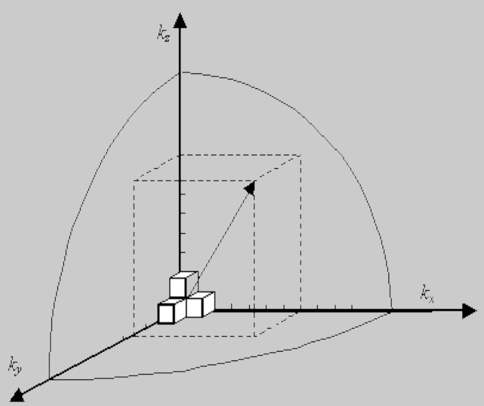
The "ultraviolet catastrophe"

- The classical prediction for black body radiation did fine for low frequencies, but failed at higher frequencies (e.g. UV)
- The problem is that the number of allowed modes increases dramatically as the wavelength gets shorter.
- Need to:
  - Calculate the "density of states" as f(ω)
  - Weight the probability of excitation correctly.



Density of states

- For a given box size, there is a low frequency cutoff but no cutoff for high frequencies
- Near a given frequency, there will be a number of combinations of k's *l,m,n* for that frequency



$$N(k) = \text{\#pol states} \times \frac{\text{volume of k-space octant}}{\text{volume of unit k-space cell}}$$

$$= 2\frac{\frac{1}{8}(4/3)\pi k^3}{\frac{\pi}{L_x}\times\frac{\pi}{L_y}\times\frac{\pi}{L_z}} = \frac{k^3}{3\pi^2}V$$

Density of modes = density of states $\quad g(k)\,dk = \frac{1}{V}\frac{dN(k)}{dk}dk = \frac{k^2}{\pi^2}dk$

Other forms: $\quad g(\omega)\,d\omega = \frac{\omega^2}{\pi^2 c^3}d\omega \quad g(\nu)\,d\nu = 8\pi\frac{\nu^2}{c^3}d\nu$

Spectral energy density

- Generalize EM energy density to allow for spectral distribution

$$\rho(\nu)d\nu = \text{excitation energy per mode} \times \text{density of modes}$$

  - Total energy density: $\int \rho(\nu)d\nu$
  - Classical form:

$$\rho(\nu)d\nu = k_B T \, \frac{8\pi\nu^2}{c^3} d\nu$$

  - Problem: total energy is infinite!

- Planck: only allow quantized energies for each mode

$\mathcal{E} = \left(n + \frac{1}{2}\right)h\nu$ n = number of photons in each mode

  - Now get average energy/mode with sum, not integral

$$P_n = \frac{e^{-\mathcal{E}_n/k_B T}}{\sum_j e^{-\mathcal{E}_j/k_B T}}$$

Mean photon number: $\bar{n} = \sum_n n P_n$

Blackbody spectrum

- Mean number of photons per mode:

$$\bar{n} = \sum_j n P_n = 1 \Big/ \left(e^{h\nu/k_B T} - 1\right)$$

- Spectral energy density of BB radiation:

$$\rho(\nu)\,d\nu = \text{avg \# photons per mode} \times h\nu \text{ per photon} \times \text{density of modes}$$

$$= \frac{1}{e^{h\nu/k_B T} - 1} h\nu \; g(\nu)\,d\nu = 8\pi \frac{\nu^2}{c^3} \frac{h\nu}{e^{h\nu/k_B T} - 1} d\nu$$