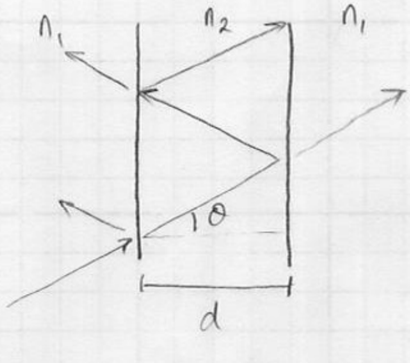
The Fabry-Perot interferometer and multiple-beam interference

A major component of modern optical technology is the Fabry-Perot interferometer. It's basically two reflective surfaces with a gap in between. Let's analyze it formally and see what makes it so special.



Let's model things as two interfaces of some reflectivity. There's light coming in from the left, and some fraction gets reflected back to the left. Some fraction makes it into the gap. And some fraction makes it all the way through. We can even throw in the complication of non-normal incidence.

We've solved problems like this before using wave mechanics. You write the net field in each region and apply boundary conditions to find ratios like E_R/E_I. It's not too bad, and you don't even need to explicitly treat all the multiple reflections. However, in the interest of learning new techniques, let's try a ray optics approach.

First we need to derive some results known as the Stokes relations, which are a consequence of something called the Principle of Reversibility.

Put simply, ray optics is time-reversable. If light rays can follow a certain path going left to right, it must also be able to go right to left, with the same relative amplitudes.

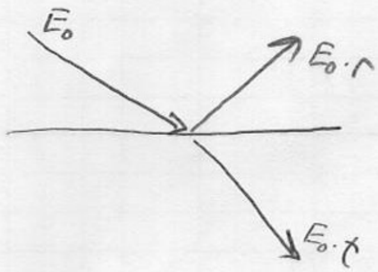
If light can do this:

It can also do this:

And if light can do this:

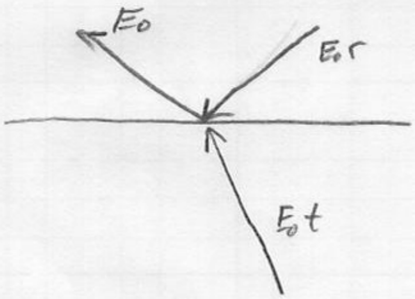
It can also do this:

Things get weird when you consider that light can do this:

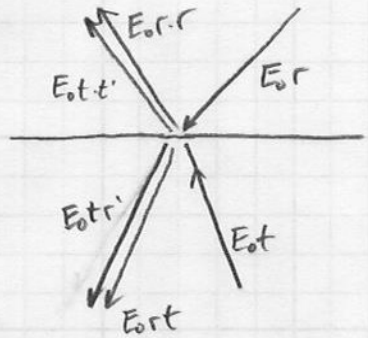


Where E_0 is the incident field amplitude, r is the fraction of the field amplitude that gets reflected, and t is the transmitted fraction. Note that since these refer to fields and not intensities, $r + t \neq 1$.

If light can do that, reversibility demands that it can do this:



But wait... if this is an interface where incoming light gets partly reflected and partly transmitted (which it is, judging from the first case), then part of the $E_0 r$ ray has to get transmitted (some fraction t of it, for a total of $E_0 r t$). And part of $E_0 t$ has to get reflected. And part of $E_0 t$ gets transmitted, and part of $E_0 r$ has to get reflected.



So given $E_o r$ and $E_o t$ coming in from the right, we have to have those four other rays too, with the indicated amplitudes. Note that we're distinguishing between t, the transmitted fraction as we go from the top material to the bottom, and t', the fraction as we go from bottom material to top. It may seem like these have to be the same, but if we're dealing with fields instead of intensities, going one way vs. the other might differ by a phase shift or something.

Similarly, we distinguish between r and r'.

Now, comparing to the previous, reversibility demands that $E_o r \cdot r$ and $E_o t \cdot t'$ combine to give E_o, and $E_o t r'$ and $E_o r t$ combine to give zero.

$$E_o t t' + E_o r r = E_o \quad \Rightarrow \quad t t' + r r = 1$$

$$\Rightarrow \quad \boxed{t t' = 1 - r^2}$$

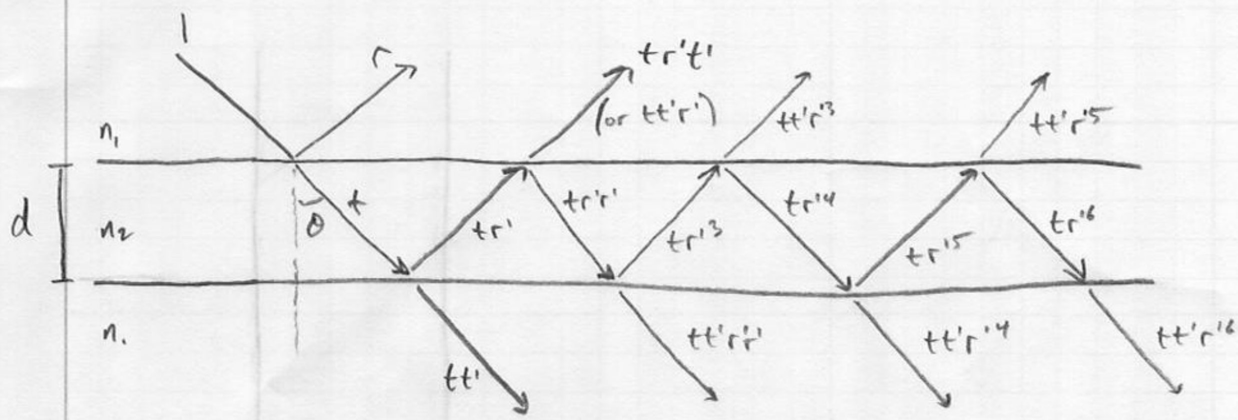
$$\text{and} \qquad E_o t r' + E_o r t = 0 \quad \Rightarrow \quad \boxed{r' = -r}$$

These relationships are called the <u>Stokes relations</u>.

$r' = -r$ we actually already knew: This is expressing the fact that reflections going one way (from low index to high index, for example) will be phase shifted by 180° relative to reflections going the other way (from high index to low index).

We derived this with wave optics before. I find it interesting that you can also get it with ray optics, a much more primitive theory.

What of $t t' = 1 - r^2$? What on earth could <u>that</u> be good for? Better file it away as we return to the Fabry-Perot.

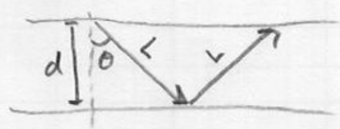


Here's the Fabry Perot again, turned on its side, with all the multiple reflections tracked. Light of unit amplitude comes in from the upper left. Every time it reflects going from n_1 to n_2, the ray picks up a factor of r. Every time it reflects going from n_2 to n_1, it picks up r'. Transmitting from n_1 to n_2 picks up a factor of t, and from n_2 to n_1 picks up t'.

The net light reflected from the top is the sum of all those individual rays, each of which has undergone some number of reflections + transmissions, losing some fraction of the field each time.

But wait, it gets worse! As the light bounces around, covering distance, it accumulates phase. After traveling some Δx, a ray will have some attached phase factor $e^{ik\Delta x}$, where $k = 2\pi/\lambda$, λ being the wavelength in whatever material the ray is traveling through.

Let's look at a special case. Let's let the gap separation d be such that for our combination of θ and λ, the round trip (top to bottom, back to top) phase accumulated is 2π:



$L = \frac{d}{\cos\theta}$ $\quad \Delta x = \frac{2d}{\cos\theta}$ $\quad \lambda, d, \theta$ such that $e^{ik\Delta x} = e^{i2\pi} = 1$

Then we can drop all consideration of distance-related phase and figure out how much light is reflected by summing the rays.

We have $\frac{E_{reflected,net}}{E_{incoming}} = r + tt'r' + tt'r'^3 + tt'r'^5$

$$= r + tt'r'\left(1 + r'^2 + r'^4 + \dots\right)$$

$$= r + tt'r' \sum_{n=0}^{\infty} (r'^2)^n$$

That series is a geometric series that converges since $r'^2 < 1$,
so $\sum_{n=0}^{\infty} (r'^2)^n = \frac{1}{1-r'^2}$ and

$$\frac{E_{ref}}{E_{inc}} = r + tt'r' \cdot \frac{1}{1-r'^2}$$

And a Stokes relation yields $tt' = 1 - r^2$

$$= r + r'\frac{\cancel{(1-r^2)}}{\cancel{(1-r'^2)}}$$

And if $r = -r'$, $r^2 = r'^2$

$$= r + r' = 0$$

<u>No</u> reflected wave. That means its <u>all</u> transmitted, and that happens irrespective of r. Even if the boundary is <u>extremely reflective</u>, interference will conspire to make sure that this special wavelength satisfying $\frac{2d}{\cos\theta} \cdot k = 2m\pi$

will pass through the Fabry-Perot <u>like it isn't even there</u>.

That's kind of special. Maybe we're going to be able to use this thing like some kind of near-perfect filter, without needing a fancy multilayer dielectric coating.

Now lets do the general case, where the plate separation and angle of incidence could be anything.

As a ray bounces in between the plates, each trip causes it to accumulate some additional phase $e^{ik\Delta x}$

with $\Delta x = \frac{2d}{\cos\theta}$ (round trip)

so each of the rays on the reflected (or transmitted) side picks up an additional factor of $e^{ik\Delta x}$,

which we'll shorthand as $e^{i\delta}$.

The sum yielding the reflected field amplitude therefore becomes:

$$\frac{E_{\text{reflected}}}{E_{\text{incoming}}} = r + tt'r'e^{i\delta} + tt'r'^3 e^{2i\delta} + tt'r'^5 (e^{i\delta})^3$$

$$= r + tt'r'e^{i\delta} \sum_{n=0}^{\infty} \left(r'^2 e^{i\delta}\right)^n$$

$$= r + tt'r'e^{i\delta} \frac{1}{\left(1 - r'^2 e^{i\delta}\right)}$$

$$= r - \frac{tt' r e^{i\delta}}{1 - r'^2 e^{i\delta}} \qquad (r = -r')$$

$$= r - \frac{(1-r^2) r e^{i\delta}}{1 - r^2 e^{i\delta}} \qquad (tt' = 1 - r^2, \ r^2 = r'^2)$$

$$= r\left[1 - \frac{(1-r^2)e^{i\delta}}{1 - r^2 e^{i\delta}}\right]$$

$$= r\left[\frac{1 - \cancel{r^2 e^{i\delta}}}{1 - r^2 e^{i\delta}} - \frac{e^{i\delta}}{1 - r^2 e^{i\delta}} + \frac{\cancel{r^2 e^{i\delta}}}{1 - r^2 e^{i\delta}}\right]$$

$$= r\left[\frac{1 - e^{i\delta}}{1 - r^2 e^{i\delta}}\right]$$

That's complex and not super transparent, so let's find the reflection coefficient big R from the intensities

$$R \equiv \frac{I_{\text{ref}}}{I_{\text{incident}}} = \frac{|E_{\text{ref}}|^2}{|E_{\text{inc}}|^2}$$

$$= r^2 \left[\frac{1 - e^{i\delta}}{1 - r^2 e^{i\delta}}\right]\left[\frac{1 - e^{-i\delta}}{1 - r^2 e^{-i\delta}}\right] = r^2 \left[\frac{1 - e^{i\delta} - e^{-i\delta} + 1}{1 - r^2 e^{i\delta} - r^2 e^{-i\delta} + r^4}\right]$$

$$= r^2 \left[\frac{2 - 2\left[\frac{e^{i\delta} + e^{-i\delta}}{2}\right]}{1 + r^4 - 2r^2\left[\frac{e^{i\delta} + e^{-i\delta}}{2}\right]}\right]$$

$$= r^2 \left[\frac{2 - 2\cos\delta}{1 + r^4 - 2r^2 \cos\delta}\right]$$

And since $1-\cos 2\delta = 2\sin^2\delta$, $\quad 2-2\cos\delta = 4\sin^2\delta/2$

Also, for the denominator, $\quad \cos\delta = 1-2\sin^2\delta/2$

$$\Rightarrow \quad R = 4r^2\left[\frac{\sin^2\delta/2}{1+r^4-2r^2+4r^2}\right]$$

$$= 4r^2\left[\frac{\sin^2\delta/2}{(1-r^2)^2+4r^2\sin^2\delta/2}\right]$$

That's pure real, but could still be easier to interpret.

Let's define something called the finesse:

$$\boxed{F \equiv \left[\frac{2r}{(1-r^2)}\right]^2}$$

If you do that, you can rewrite the above as:

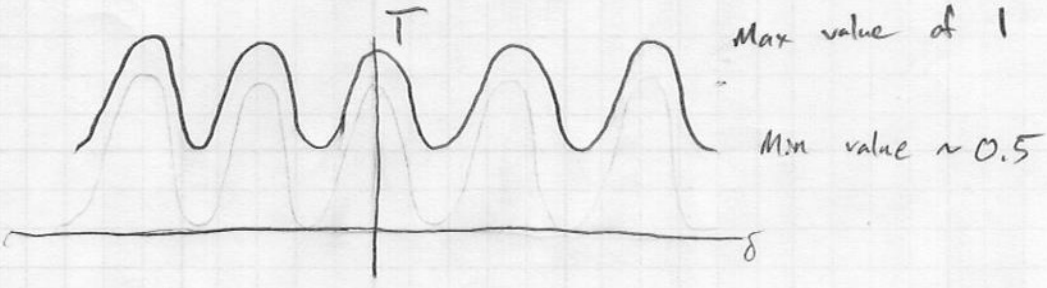
$$\boxed{R = \frac{F\sin^2\delta/2}{1+F\sin^2\delta/2}}$$

And now we're in a position to do some interpretation.

r is some number between 0 and 1.
As r approaches 1, $F\to\infty$ (and as $r\to 0$, $F\to 0$)

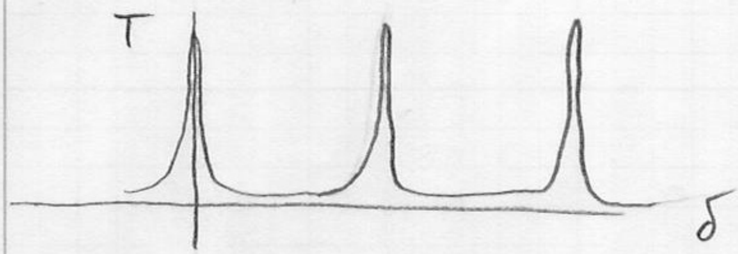
For any particular value of F, R has a numerator that oscillates between 0 and F periodically, and a denominator that oscillates with the same period.

Let's look at some sketches of T, which comes from $1-R$
For $F=1$, we'd get something like:

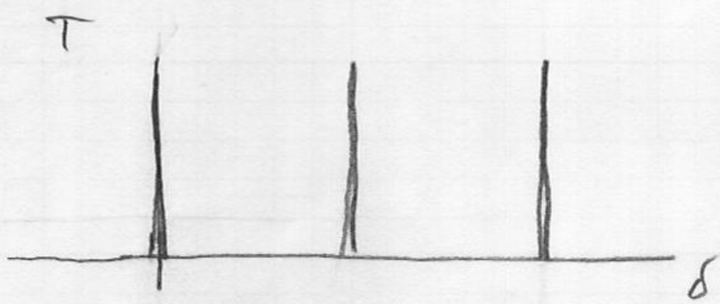


The transmission goes to 1 whenever $\delta/2$ is $m\pi$, and in between it drops off, though never quite to zero.

For a finesse of 100, the graph looks like:



And for a finesse of 10000, we get:



Something with really sharp peaks. Certain frequencies are passed perfectly, while all others are wiped out almost completely. It's one hell of a filter.