4/13/08 Review

Note Title 4/13/2008

Concepts

wave functions live in Hilbert space

[with a few notable hard cases such as $e^{ipx/\hbar}$]

[E.g. the set of square integrable functions on [a,b]]

inner products

$$|\alpha\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix} \qquad |\beta\rangle = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix}$$

this N-tuple represents $|\beta\rangle$

$$\langle\beta|\alpha\rangle = b_1^* a_1 + b_2^* a_2 + \cdots$$

$$= b_i^* a_i \qquad \text{summation conv.}$$

$$|f\rangle = f(x) \qquad |g\rangle = g(x)$$

$$\langle f|g\rangle = \int_a^b f^*(x)\, g(x)\, dx$$

$$\langle g|f\rangle^* = \langle f|g\rangle$$

$$\langle f|f\rangle = \int_a^b |f|^2\, dx$$

A complete, orthonormal set of functions f_n satisfies

a) for ANY $f(x)$ in Hilbert space

$$f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$$

b) $\langle f_n | f_m \rangle = \delta_{mn}$ $\rightarrow$ this implies both orthog. + normalization

NB if $f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$ then

$$c_n = \langle f_n | f \rangle$$

Hermitian operators + observables

$Q(x,p)$ observable, expectation $\langle Q \rangle$

$$\equiv \int \psi^* \hat{Q} \psi \, dx = \langle \psi | \hat{Q} \psi \rangle$$

Outcome of any msmt is real so

$$\langle Q \rangle = \langle Q \rangle^*$$

or $\langle \psi | \hat{Q} \psi \rangle = \langle \psi | \hat{Q} \psi \rangle^*$

$$= \langle \hat{Q} \psi | \psi \rangle$$

$$\int \psi^* \hat{A} \psi \, dx$$

$$= \int (A^{+} \psi^*) \psi \, dx$$

This is because of the complex conjugate in $\langle | \rangle$.

Hence $\hat{Q}$ must be Hermitian.

observables are represented by Hermitian operators

green means potential test prob

Prove that p is Hermitian in coordinate representation

Problem 3.4

Let Q be an observable.
If $\sigma_Q^2 = 0$ the state is said to be determinant.

Show that $\sigma^2 = 0 \Rightarrow \hat{Q}\psi = q\psi$ where $\langle \hat{Q} \rangle = q$.

The collection of all eigen values of an operator is its spectrum

Compute the spectrum of $\hat{Q} = \frac{d^2}{d\phi^2}$.

Assuming the E-functions are

periodic with period 2π.

If 2 or more Linearly independent eigenfunctions share the same eigenvalue, the spectrum is degenerate

Discrete spectra
Continuous spectra
mixed.

$\boxed{\text{Axiom: The eigenfunctions of an observable operator are complete.}}$

if you measure an observable $Q(x,p)$ on a particle in a state $\Psi(x,t)$, then you are certain to get an eigenvalue of the Hermit. operator $\hat{Q}(x, -i\hbar\frac{d}{dx})$

in the case of discrete spectra the prob. of measuring the E-value q_n associated with $f_n(x)$ is

$$|c_n|^2, \text{ where } c_n = \langle f_n | \Psi \rangle$$

for continuous spectra, the e-value is a function $q(z)$. the prob. becomes

$$|c(z)|^2 \, dz \quad \text{where} \quad c(z) = \langle f_z | \Psi \rangle$$

Read footnote 16 on page 107

$$\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \, \Psi(x,t) \, dx$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{ipx/\hbar} \, \Phi(p,t) \, dp.$$

Example 3.4

There is an uncertainty relation connecting every pair of incompatible observables:

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [A,B] \rangle \right)^2$$

e.g. $\sigma_x \sigma_p \geq \hbar/2$ or $\Delta x \Delta p \geq \hbar/2$

same thing.

But the statement that

$\Delta t \, \Delta E \geq \hbar/2$ is of a

fundamentally different nature
See the proof on page 115 – 116

Read example 3.7. Something like
this will be on the test.
Also the Footnote on p. 117.

Understand the resolution of the
identity 3.91 + 3.94

$$\sum_n |e_n\rangle\langle e_n| = 1$$

$$\int |e_z\rangle\langle e_z| \, dz = 1$$

$|\alpha\rangle\langle\beta|$
outer prod.
Dirac not.

assuming $\langle e_m | e_n \rangle = \delta_{mn}$ Kronecker delta.
or $\langle e_z | e_{z'} \rangle = \underbrace{\delta(z-z')}_{\text{Dirac orthogonality}}$

Project onto $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$ $(1, 0, 0, 0, \ldots) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$

Do problems 3.23, 3.24

$$\sum_n |e_n\rangle\langle e_n| = 1$$

$$\sum_n |e_n\rangle\langle e_n|\alpha\rangle = |\alpha\rangle$$

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \qquad \vec{e}_2 = \begin{pmatrix} 0 \\ \vdots \end{pmatrix}$$

$$\vec{e}_1 \vec{e}_1^{\,T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \\ & & \end{pmatrix} \qquad \vec{e}_2 \vec{e}_2^{\,T} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \end{pmatrix}$$

$$\sum_{\gamma=1}^{N} \vec{e}_\gamma \vec{e}_\gamma^{\,T} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \end{pmatrix}$$

$$\sum |e_\gamma\rangle\langle e_\gamma|$$

$\boxed{\text{Example 3.8}}$