3_28_08

Note Title 3/28/2008

general homogeneous 2nd order ODE

$$y'' + p(x)\, y' + q(x)\, y = 0$$

$$y' = dy/dx \qquad y'' = d^2y/dx^2$$

most general solution is

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where $y_1(x)$ + $y_2(x)$ are linearly indep.
i.e.

$$\boxed{\begin{array}{l} \text{if} \quad c_1 y_1(x) + c_2 y_2(x) = 0 \\ \text{then} \quad c_1 = c_2 = 0 \end{array}}$$

linear indep.

if $y_1 + \alpha y_2 = 0$
$\Rightarrow y_1' + \alpha y_2' = 0$

$\Rightarrow y_1 - y_1' + \alpha(y_2 - y_2') = 0$
and $(y_1 + y_1') + \alpha(y_2 + y_2') = 0$

$$\frac{y_1 - y_1'}{y_2 - y_2'} = -\alpha \Rightarrow (y_1 + y_1') = \frac{y_1 - y_1'}{y_2 - y_2'}(y_2 + y_2')$$

$$(y_1 + y_1')(y_2 - y_2') = (y_1 - y_1')(y_2 + y_2')$$

$$y_1 y_2 + y_1' y_2 - y_1 y_2' - y_1' y_2'$$

$$= y_1 y_2 - y_1' y_2 + y_1 y_2' - y_1' y_2'$$

$$\Rightarrow \quad 2y_1' y_2 - 2y_1 y_2' = 0$$

$$\Rightarrow \quad y_1' y_2 - y_1 y_2' = 0$$

or $\quad y_1 y_2' - y_2 y_1' = 0$

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

<u>Wronskian</u>

Eg. $\quad y'' + y = 0$

we know $\sin x$ + $\cos x$ are both solutions.

$$W(x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1 \neq 0$$

To solve inhomogeneous Equations.

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

1) 1st solve the homog. equation
$c_1 y_1(x) + c_2 y_2(x)$

2) find any solution of
$y_p(x)$ {particular solution}

3) general solution is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

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$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$ Standard power series

$$= a_0 + a_1 x + a_2 x^2 + \cdots$$

$$\boxed{\left.\begin{array}{l} f(0) = a_0 \\ f'(0) = a_1 \\ f''(0) = 2a_2 \\ \vdots \end{array}\right\} \quad \begin{array}{l} a_0 = f(0) \\ a_1 = f'(0) \\ a_2 = \frac{1}{2} f''(0) \\ \vdots \end{array}}$$

a) $f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$

b) $= \sum_{n=1}^{\infty} n a_n x^{n-1}$ direct translation of

shift $\quad n-1 \Rightarrow n$

$\quad n \Rightarrow n+1$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$ but we must make sure it agrees with a) or b)

$$\downarrow$$

$a_1 x^0 + 2a_2 x^1 + \cdots$ looks ok.

$$\boxed{\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n}$$

This is just like changing variables in an integration

A Diff. Eqn

$$f''(x) + f(x) = 0$$

start with $f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$

$$f'' = 2a_2 + 6a_3 x + \cdots$$

$$= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

So $f''(x) + f(x) \Rightarrow$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + a_n \right] x^n = 0$$

This must be true for arbitrary x, so we must have

$$(n+2)(n+1) a_{n+2} + a_n = 0$$

or

$$\boxed{a_{n+2} = \frac{-a_n}{(n+2)(n+1)}}$$ Recursion relation

Since the ODE is second order we expect the need to specify 2 coefficients to generate the rest

Eg. $\boxed{a_0 = 0, \quad a_1 = 1}$ This is a plausible guess!

$$\Rightarrow \quad a_2 = 0 \qquad a_3 = \frac{a_1}{3 \cdot 2} = -\frac{1}{6}$$

$$a_4 = 0 \qquad a_5 = \frac{a_3}{5 \cdot 4} = \frac{1}{5!}$$

$$\vdots$$

$$f(x) = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \cdots$$

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

On the other hand we could have* chosen $\boxed{a_0 = 1 \quad a_1 = 0}$ another guess

$$\boxed{a_{n+2} = \frac{-a_n}{(n+2)(n+1)}} \quad \text{so}$$

$$a_2 = -\frac{1}{2} \qquad a_3 = 0 \qquad a_4 = \frac{1}{4!}$$

* ultimately, we must show the solutions are

Lin. indep.

So $f(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots$

$$\boxed{f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}}$$

Look familiar?

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots$$

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots$$

So $a_0 = 0 \quad a_1 = 1$ gives

$$f(x) = \sin(x)$$

$a_0 = 1 \quad a_1 = 0$ gives

$$f(x) = \cos(x)$$

More complicated example

Bessel eqn.

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - m^2)y = 0$$

$$\Rightarrow \quad y'' + \frac{1}{x}y' + \frac{x^2 - m^2}{x^2}y = 0$$

$$y'' + \frac{1}{x}y' + y - \frac{m^2}{x^2}y = 0$$

in this case we look for solutions of the form

$$y = x^s \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2}$$

plugging in

$$\sum_{n=0} (n+s)(n+s-1) a_n x^{n+s-2} + \sum_{n=0} (n+s) a_n x^{n+s-2} + \sum_{n=0} a_n x^{n+s} - m^2 \sum_{n=0} a_n x^{n+s-2}$$

$$\sum_{n=0}^{\infty} \left[(n+s)(n+s-1) + (n+s) - m^2\right] a_n x^{n+s-2} + \sum_{n=2}^{\infty} a_{n-2} x^{n+s-2} = 0$$

$$(n+s)(n+s-1) = n^2 + 2ns - n + s^2 - s$$

so $(n+s)(n+s-1) + (n+s) = (n+s)^2$

$$\sum_{n=0}^{\infty} \left[(n+s)^2 - m^2\right] a_n x^{n+s-2} + \sum_{n=2}^{\infty} a_{n-2} x^{n+s-2} = 0$$

coefficient of x^{s-2}

$$(s^2 - m^2)\, a_0 = 0$$

$a_0 \neq 0 \Rightarrow \quad s = \pm m$

$s_1 = m \qquad s_2 = -m$

plug $s = m$ into

$$\sum_{n=0}^{\infty} \left[(n+s)^2 - m^2\right] a_n x^{n+s-2} + \sum_{n=2}^{\infty} a_{n-2} x^{n+s-2} = 0$$

$\uparrow$

we took care of $n=0$

$$\Rightarrow \sum_{n=1}^{\infty} n(n+2m)\, a_n x^{n+m-2} + \sum_{n=2}^{\infty} a_{n-2} x^{n+m-2} = 0$$

continue comparing coefficients.

$n=1$) $\quad x^{m-1}: \quad (2m+1)\, a_1 = 0$

$\Rightarrow a_1 = 0$

$n \geq 2 \quad x^{m+n-2}: \; n(n+2m)\, a_n + a_{n-2} = 0$

$$\Rightarrow a_n = -\frac{a_{n-2}}{n(n+2m)}$$

since $a_1 = 0$ $a_n = 0$ for odd n.

for $n = 2k$

$$a_{2k} = \frac{-1}{2k(2k+2m)} a_{2k-2}$$

$$= \frac{-1}{4k(k+m)} a_{2(k-1)}$$

$\vdots$

iterate this down to a_0

$$a_{2k} = \frac{-1}{4k(k+m)} \frac{-1}{4(k-1)(m+k-1)} \cdots \frac{-1}{k(m+1)} a_0$$

$$\boxed{a_{2k} = \frac{(-1)^k m!}{2^{2k} k! (m+k)!} a_0}$$

So putting this all together

$$y(x) = \sum_{k=0}^{\infty} a_{2k} x^{2k+m}$$

$$= 2^m m!\, a_0 \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k+m}}{k!\,(m+k)!}$$

$$= 2^m m!\, a_0 \; J_m(x)$$

where $J_m(x)$.