

P2 Solutions

1) a)  $Z = 1 + e^{-\epsilon/\tau}$

$F = -\tau \log Z = -\tau \log(1 + e^{-\epsilon/\tau})$

b)  $\sigma = -\left(\frac{\partial F}{\partial \tau}\right)_V = \log(1 + e^{-\epsilon/\tau}) + \frac{\tau}{1 + e^{-\epsilon/\tau}} \left(\frac{+\epsilon}{\tau^2} e^{-\epsilon/\tau}\right)$

$\sigma = \log(1 + e^{-\epsilon/\tau}) + \frac{\epsilon/\tau}{e^{\epsilon/\tau} + 1}$

$F = U - \tau \sigma$

$U = F + \tau \sigma = \frac{\epsilon}{e^{\epsilon/\tau} + 1}$

2) a)  $Z = (Z_1)^N = (e^{mB/\tau} + e^{-mB/\tau})^N$

$= [2 \cosh(mB/\tau)]^N = \sum_{s=-N/2}^{N/2} e^{mB 2s/\tau}$

$M = +\frac{m}{V} \langle 2s \rangle = -\tau^2 \left[ \frac{+\partial(Z)}{\partial \tau} \right]_{BVZ} = -\frac{\tau^2}{BV} \frac{\partial(\log Z)}{\partial \tau}$

$\therefore -\frac{\tau^2}{BV} \frac{\partial}{\partial \tau} N \log[2 \cosh(mB/\tau)] = \frac{N}{BV} (+mB \tanh(mB/\tau))$

$\therefore M = +nm \tanh(mB/\tau)$  [Note:  $|M| \leq nm$ ]

$\chi = \frac{dM}{dB} = +\frac{nm^2}{\tau} \frac{1}{\cosh^2(mB/\tau)}$

b)  $F = -\tau \log Z = -N\tau \log[2 \cosh(mB/\tau)]$

$\tanh(mB/\tau) = \frac{-M}{nm} \equiv -x$

set  $y = e^{mB/\tau}$

$\frac{y^2 - 1}{y^2 + 1} = -x$

$(1+x)y^2 = 1-x$

$y = \pm \left(\frac{1-x}{1+x}\right)^{1/2}$  ( $|x| \leq 1$ )

$F = -N\tau [\log(y^2 + 1) - \log y] = -N\tau \log \left[ \frac{2}{1+x} \right] + \frac{N\tau}{2} \log \left( \frac{1-x}{1+x} \right)$

$$F = -N\tau \log \left( \frac{2}{\sqrt{1-x^2}} \right)$$

$$c) \quad \chi = +nm^2 \frac{1}{\tau \cosh^2(m\epsilon/\tau)} \approx \frac{nm^2}{\tau}$$

$\uparrow$   
 $m\epsilon/\tau \ll 1$

$$3) \quad a) \quad Z = \sum_{s=0}^{\infty} e^{-sth\omega/\tau} = \frac{1}{1 - e^{-h\omega/\tau}}$$

$$F = -\tau \log Z = +\tau \log(1 - e^{-h\omega/\tau})$$

$$b) \quad \sigma = -\left(\frac{\partial F}{\partial \tau}\right)_{\nu} = -\log(1 - e^{-h\omega/\tau}) - \tau \frac{(-h\omega/\tau^2) e^{-h\omega/\tau}}{1 - e^{-h\omega/\tau}}$$

$$= -\log(1 - e^{-h\omega/\tau}) + \frac{h\omega/\tau}{e^{h\omega/\tau} - 1}$$

$$4) \quad U = \sum_s \epsilon_s P(\epsilon_s) = \sum_s \frac{\epsilon_s e^{-\epsilon_s/\tau}}{\sum_s e^{-\epsilon_s/\tau}} = \langle \epsilon \rangle = \frac{\left(\frac{\partial}{\partial \lambda} Z\right)_{\nu}}{Z}$$

$$\langle \epsilon^2 \rangle = \sum_s \epsilon_s^2 P(\epsilon_s) = \sum_s \frac{\epsilon_s^2 e^{-\epsilon_s/\tau}}{\sum_s e^{-\epsilon_s/\tau}} = \frac{1}{Z} \left(\frac{\partial^2 Z}{\partial \lambda^2}\right)_{\nu}$$

$$\langle (\epsilon - \langle \epsilon \rangle)^2 \rangle = \langle \epsilon^2 \rangle - 2\langle \epsilon \rangle^2 + \langle \epsilon \rangle^2$$

$$= \langle \epsilon^2 \rangle - \langle \epsilon \rangle^2$$

$$\therefore \langle (\epsilon - \langle \epsilon \rangle)^2 \rangle = \sum_s \epsilon_s^2 P(\epsilon_s) - \left(\sum_s \epsilon_s P(\epsilon_s)\right)^2$$

$$= \left[ \frac{1}{Z} \frac{\partial^2 Z}{\partial \lambda^2} - \left(\frac{1}{Z} \frac{\partial Z}{\partial \lambda}\right)^2 \right]_{\nu}$$

$$= \frac{\partial^2}{\partial \lambda^2} (\log Z) \Big|_{\nu} = -\left(\frac{\partial U}{\partial \tau}\right)_{\nu} = \tau^2 \left(\frac{\partial U}{\partial \tau}\right)_{\nu}$$

8. Ground orbital :  $n_x = n_y = n_z = 1$

$$E_0 = \frac{\hbar^2}{2M} \left(\frac{\pi}{L}\right)^2 3 = \text{energy of ground state}$$

set

$$\frac{3\hbar^2\pi^2}{2M} L^{-2} = \tau \Rightarrow L^{-3} = \left(\frac{2M\tau}{3\hbar^2\pi^2}\right)^{3/2} = \left(\frac{4}{3\pi}\right)^{3/2} \left(\frac{M\tau}{2\pi\hbar^2}\right)^{3/2}$$

$$= \left(\frac{4}{3\pi}\right)^{3/2} n_Q = 0.28 n_Q$$

↑  
number of orders 1