Maxwell's Equations to wave eqn

- Write Maxwell's eqns for a linear medium

$$\vec{\nabla}\cdot\mathbf{D}=\vec{\nabla}\cdot\left(\varepsilon_0\varepsilon\mathbf{E}\right)=0 \qquad \vec{\nabla}\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}=-\mu_0\mu\frac{\partial\mathbf{H}}{\partial t}$$

$$\vec{\nabla}\cdot\mathbf{B}=\vec{\nabla}\cdot\left(\mu_0\mu\mathbf{H}\right)=0 \qquad \vec{\nabla}\times\mathbf{H}=\frac{\partial\mathbf{D}}{\partial t}=\varepsilon_0\varepsilon\frac{\partial\mathbf{E}}{\partial t}$$

- Assume:
  - Non-magnetic medium ($\mu = 0$)
  - Linear medium $D = \varepsilon_0\, \varepsilon\, \mathbf{E}$
  - Non-dispersive medium

Take the curl:

$$\vec{\nabla}\times\left(\vec{\nabla}\times\mathbf{E}\right)=-\mu_0\frac{\partial}{\partial t}\vec{\nabla}\times\mathbf{H}=-\mu_0\frac{\partial}{\partial t}\left(\varepsilon_0\varepsilon\frac{\partial\mathbf{E}}{\partial t}\right)=-\frac{1}{c^2}\frac{\partial}{\partial t}\left(\varepsilon\frac{\partial\mathbf{E}}{\partial t}\right)$$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\mathbf{E}\right)=\vec{\nabla}\left(\vec{\nabla}\cdot\mathbf{E}\right)-\left(\vec{\nabla}\cdot\vec{\nabla}\right)\mathbf{E}$$

BAC-CAB vector ID

EM wave equation for spatially uniform media

- Generalized wave equation

$$\vec{\nabla}\left(\vec{\nabla}\cdot\mathbf{E}\right)-\left(\vec{\nabla}\cdot\vec{\nabla}\right)\mathbf{E}=-\frac{1}{c^2}\frac{\partial}{\partial t}\left(\varepsilon\frac{\partial\mathbf{E}}{\partial t}\right)=-\frac{1}{c^2}\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2}$$

If ε is time-independent

  - If medium has a spatially-uniform refractive index:

$$\vec{\nabla}\cdot\left(\varepsilon\mathbf{E}\right)=\varepsilon\vec{\nabla}\cdot\mathbf{E}+\left(\mathbf{E}\cdot\vec{\nabla}\right)\varepsilon=0$$

$$\vec{\nabla}^2\mathbf{E}-\frac{1}{c^2}\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2}=0 \qquad \varepsilon=n^2$$

  - If the medium is spatially varying, then for P polarized light (where E has component along gradient), then

$$\vec{\nabla}^2\mathbf{E}+\vec{\nabla}\left(\left(\mathbf{E}\cdot\vec{\nabla}\right)\ln\varepsilon\right)-\frac{1}{c^2}\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2}=0$$

3D EM wave propagation

$$\nabla^2\mathbf{E}-\frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E}=\frac{\partial^2}{\partial z^2}\mathbf{E}+\nabla_\perp{}^2\mathbf{E}-\frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E}=0$$

$$\boxed{\nabla_\perp^2=\partial_x^2+\partial_y^2}\qquad\boxed{\nabla_\perp^2=\frac{1}{r}\partial_r\left(r\,\partial_r\right)+\frac{1}{r^2}\partial_\phi^2}$$

- Note:
  - All linear propagation effects are included in LHS: diffraction, interference, focusing…
  - With plane waves transverse derivatives are zero.
- More general examples:
  - Gaussian beams (including high-order)
  - Waveguides
  - Arbitrary propagation
  - Can determine discrete solutions to linear equation (e.g. Gaussian modes, waveguide modes), then express fields in terms of those solutions.

General 3D plane wave solution

- Assume separable function

$$\mathbf{E}(x,y,z,t) \sim f_1(x) f_2(y) f_3(z) g(t)$$

$$\vec{\nabla}^2 \mathbf{E}(z,t) = \frac{\partial^2}{\partial x^2}\mathbf{E}(z,t) + \frac{\partial^2}{\partial y^2}\mathbf{E}(z,t) + \frac{\partial^2}{\partial z^2}\mathbf{E}(z,t) = \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E}(z,t)$$

- Solution takes the form:

$$\mathbf{E}(x,y,z,t) = \mathbf{E_0} e^{ik_x x} e^{ik_y y} e^{ik_z z} e^{-i\omega t} = \mathbf{E_0} e^{i\left(k_x x + k_y y + k_z z\right)} e^{-i\omega t}$$

$$\mathbf{E}(x,y,z,t) = \mathbf{E_0} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

  - Now k-vector can point in arbitrary direction

- With this solution in W.E.:

$$\boxed{n^2 \frac{\omega^2}{c^2} = k_x^2 + k_y^2 + k_z^2 = \mathbf{k}\cdot\mathbf{k}}$$

Valid even in waveguides and resonators

Grad and curl of 3D plane waves

- Simple trick:

$$\nabla \cdot \mathbf{E} = \partial_x E_x + \partial_y E_y + \partial_z E_z$$

  - For a plane wave, $\mathbf{E}(x,y,z,t) = \mathbf{E_0} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}$

$$\nabla \cdot \mathbf{E} = i\left(k_x E_x + k_y E_y + k_z E_z\right) = i\left(\mathbf{k} \cdot \mathbf{E}\right)$$

  - Similarly,

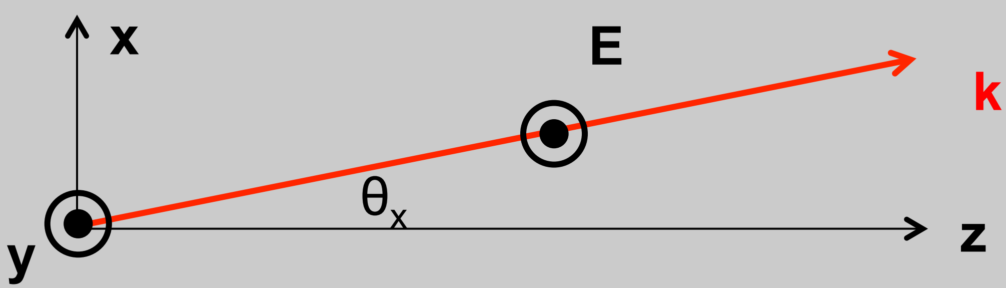
$$\nabla \times \mathbf{E} = i\left(\mathbf{k} \times \mathbf{E}\right)$$

- Consequence: since $\nabla \cdot \mathbf{E} = 0, \quad \mathbf{k} \perp \mathbf{E}$
  - For a given k direction, E lies in a plane
  - E.g. x and y linear polarization for a wave propagating in z direction

Class exercise

- Write an expression for the real *E*-field of a wave propagating in the x-z plane at an angle θ_x to the z-axis. The field is polarized in the y-direction and has an amplitude E_0. Make a sketch showing ***k*** and ***E*** relative to the coordinate system.

$$\mathbf{E}(\mathbf{r},t) = \hat{\mathbf{y}}\, E_0 \cos\left(k\, x \sin\theta_x + k\, z \cos\theta_x - \omega t\right)$$



$$k = \left|\mathbf{k}\right|$$

At t = 0 and **r** = 0

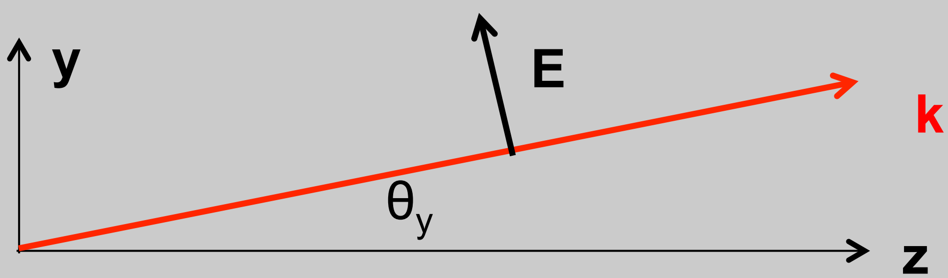
$$\mathbf{E}(0,0) = \hat{\mathbf{y}}\, E_0$$

So draw **E** vector in **+y** direction

Class exercise

- Write an expression for the complex *E*-field of a wave propagating in the y-z plane at an angle θ_y to the z-axis. The field is polarized in the y-z plane and the absolute phase is π/3, and has an amplitude E_0. Make a sketch showing ***k*** and ***E*** relative to the coordinate system.

$$\mathbf{E}(\mathbf{r},t) = E_0\left(\hat{\mathbf{y}}\cos\theta_y - \hat{\mathbf{z}}\sin\theta_y\right)\mathrm{e}^{i\left(k\,y\sin\theta_y + k\,z\cos\theta_y - \omega t + \pi/3\right)}$$



Here E_0 is real, but we will often combine the absolute phase shift with the field strength, e.g.

$$E_0 = A_0 e^{i\phi}$$