Advanced Engineering Mathematics		**Homework Six**
Fourier Transform : Sine/Cosine Transforms, Common Transforms, Convolution and Green's Functions		
Text: 11.7-11.9	Lecture Notes : 11-12	Lecture Slides: 5

Quote of Homework Six
Seems like everybody's out to test ya 'til they see your break
The Gorillaz - Dirty Harry (2005)

1. Fourier Transforms of Symmetric Functions

Let,

$$f_c(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}(\omega)\cos(\omega x)d\omega \qquad \hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(\omega x)dx \tag{1.1}$$

$$f_s(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}(\omega)\sin(\omega x)d\omega \qquad \hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(\omega x)dx, \tag{1.2}$$

be the definitions for the Fourier cosine and Fourier sine transform pairs, receptively.

1.1. **Symmetry.** Show that $f_c(x)$ and $\hat{f}_c(\omega)$ are even functions and that $f_s(x)$ and $\hat{f}_s(\omega)$ are odd functions.[1]

1.2. **Derivation from Fourier Transform.** Recall the complex Fourier transform,

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}d\omega, \quad \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx. \tag{1.3}$$

Show that if we assume that $f(x)$ is an even function then (1.3) defines the transform pair given by (1.1). Also, show that if $f(x)$ is an odd function then (1.3) defines the transform pair given by (1.2). [2]

1.3. **Even and Odd Finite Pulses.** Given,

$$f(x) = \begin{cases} A, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}, \qquad A, a \in \mathbb{R}^+. \tag{1.4}$$

Plot the even and odd extensions of f.

1.4. **Symmetric Transforms.** Find the Fourier cosine and sine transforms of f.

1.5. **Integral Trick.** Using the Fourier cosine transform show that $\displaystyle\int_{-\infty}^{\infty}\frac{\sin(\pi\omega)}{\pi\omega}d\omega = 1.$

2. Sine and Cosine Transforms

Calculate the following Fourier sine/cosine transformations. Include any domain restrictions.

2.1. **Forward Cosine Transform.** $\mathfrak{F}_c(e^{-ax}),\ \ a \in \mathbb{R}^+$

2.2. **Inverse Cosine Transform.** $\mathfrak{F}_c^{-1}\left(\dfrac{1}{1+\omega^2}\right)$

2.3. **Forward Sine Transform.** $\mathfrak{F}_s(e^{-ax}),\ \ a \in \mathbb{R}^+$

2.4. **Inverse Sine Transform.** $\mathfrak{F}_s^{-1}\left(\sqrt{\dfrac{2}{\pi}}\dfrac{\omega}{a^2+\omega^2}\right),\ \ a \in \mathbb{R}^+$

[1] Thus, if an input function has an even or odd symmetry then the transformed function shares the same symmetry.

[2] Thus, if an input function has symmetry then the Fourier transform is real-valued and reduced to a since or cosine transform.

3. Fourier Transforms

Calculate the following transforms. Include any domain restrictions.

3.1. **Dirac-Delta.** $\mathfrak{F}\{f\}$ where $f(x) = \delta(x - x_0),\ \ x_0 \in \mathbb{R}$.[3]

3.2. **Decaying Exponential Function.** $\mathfrak{F}\{f\}$ where $f(x) = e^{-k_0|x|},\ \ k_0 \in \mathbb{R}^+$.

3.3. **Even Combination of Dirac-Deltas.** $\mathfrak{F}^{-1}\left\{\hat{f}\right\}$ where $\hat{f}(\omega) = \frac{1}{2}\left(\delta(\omega + \omega_0) + \delta(\omega - \omega_0)\right),\ \ \omega_0 \in \mathbb{R}$.

3.4. **Odd Combination of Dirac-Deltas.** $\mathfrak{F}^{-1}\left\{\hat{f}\right\}$ where $\hat{f}(\omega) = \frac{1}{2}\left(\delta(\omega + \omega_0) - \delta(\omega - \omega_0)\right),\ \ \omega_0 \in \mathbb{R}$.

3.5. **Horizontal Shifts.** Find $\hat{f}(\omega)$ where $f(x + c)$, $c \in \mathbb{R}$.

4. Convolution Integrals

The convolution h of two functions f and g is defined as[4],

$$h(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(p)g(x - p)dp = \int_{-\infty}^{\infty} f(x - p)g(p)dp. \tag{4.1}$$

4.1. **Fourier Transforms of Convolutions.** Show that $\mathfrak{F}\{f * g\} = \sqrt{2\pi}\mathfrak{F}\{f\}\mathfrak{F}\{g\}$. [5]

4.2. **Simple Convolution.** Find the convolution $h(x) = (f * g)(x)$ where $f(x) = \delta(x - x_0)$ and $g(x) = e^{-x}$.

5. Simple Green's Functions

Given the ODE,

$$y' + y = f(x), \quad -\infty < x < \infty. \tag{5.1}$$

5.1. **Delta Forcing.** Calculate the frequency response, or what is sometimes called the steady-state transfer function, associated with (5.1). [6]

5.2. **Inversion.** Calculate the Green's function associated with (5.1). [7]

5.3. **Solutions as Convolution Integrals.** Using convolution find the steady-state solution to the (5.1).[8]

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[3]Here the δ is the so-called Dirac, or continuous, delta function. This isn't a function in the true sense of the term but instead what is called a generalized function. The details are unimportant and in this case we care only that this Dirac-delta *function* has the property $\int_{-\infty}^{\infty} \delta(x - x_0)f(x)dx = f(x_0)$. For more information on this matter consider `http://en.wikipedia.org/wiki/Dirac_delta_function`. To drive home that this *function* is strange, let me spoil the punch-line. The sampling function $f(x) = \operatorname{sinc}(ax)$ can be used as a definition for the Delta *function* as $a \to 0$. So can a bell-curve probability distribution. Yikes!

[4]Here wee keep the same notation as Kreysig pg. 523

[5]The point here is that while the Fourier transform of a linear combination is the linear combination of Fourier transforms, the Fourier transform of a product is a convolution integral. Well, at least that's something.

[6]This function is a representation of how the system responds to the most primitive force, $\delta(x)$, in the Fourier domain.

[7]The Green's function is just the inverse of the frequency response function and is a representation of how the system would like to respond to the primitive, $\delta(x)$, force in the original domain.

[8]The key point of these three steps is that, if you can determine how a linear differential equation responds to simple forcing then the general solution can be represented in terms of a convolution integral containing the Green's function. This gives us a general method for solving inhomogeneous differential equations. However, finding the Green's function for a DE is generally nontrivial.