6.5.11 (10 pts) For each natural number$k$, let A_k be a set, and for each natural number n, let $f_n : A_n \to A_{n+1}$. For example, $f_1 : A_1 \to A_2$, $f_2 : A_2 \to A_3$, $f_3 : A_3 \to A_4$, and so on.
Use mathematical induction to prove that for each natural number n with $n \geq 2$, if $f_1, f_2, \ldots, f_n$ are all bijections, then $f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1$ is a bijection and

$$(f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1)^{-1} = f_1^{-1} \circ f_2^{-1} \circ \cdots \circ f_{n-1}^{-1} \circ f_n^{-1}$$

Proof. Using a proof by induction, we first note that by theorem[1] presented in the text if f_1, f_2 are bijections then $f_2 \circ f_1$ is also a bijection and

$$(f_2 \circ f_1)^{-1} = f_1^{-1} \circ f_2^{-1}$$

Now assume that

$$(f_k \circ f_{k-1} \circ \cdots \circ f_2 \circ f_1)^{-1} = f_1^{-1} \circ f_2^{-1} \circ \cdots \circ f_{k-1}^{-1} \circ f_k^{-1}$$

and consider,

$$(f_{k+1} \circ f_k \circ f_{k-1} \circ \cdots \circ f_2 \circ f_1) = (\underbrace{f_{k+1}}_{g} \circ \underbrace{f_k \circ f_{k-1} \circ \cdots \circ f_2 \circ f_1}_{h}) = g \circ h$$

From our inductive step, we know h is a bijection and that

$$h^{-1} = f_1^{-1} \circ f_2^{-1} \circ \cdots \circ f_{k-1}^{-1} \circ f_k^{-1}$$

Then, from our basis step, we see that $g \circ h$ is a bijection and that

$$(g \circ h)^{-1} = h^{-1} \circ g^{-1} = f_1^{-1} \circ f_2^{-1} \circ \cdots \circ f_k^{-1} \circ f_{k+1}^{-1}$$

Thus, for each natural number n with $n \geq 2$, if $f_1, f_2, \ldots, f_n$ are all bijections, then $f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1$ is a bijection and

$$(f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1)^{-1} = f_1^{-1} \circ f_2^{-1} \circ \cdots \circ f_{n-1}^{-1} \circ f_n^{-1}$$

□

[1] Theorem 6.32: Let $f : A \to B$ and $g : B \to C$ be bijections. Then $g \circ f$ is a bijection and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

6.6.8 (10 pts) Let $f : S \to T$ and let A and B be subsets of S. Prove or disprove each of the following:

a) If $A \subseteq B$ then $f(A) \subseteq f(B)$.

Proof. Let $f(x) \in f(A) \Rightarrow x \in A \Rightarrow x \in B \Rightarrow f(x) \in f(B)$. Thus, $f(A) \subseteq f(B)$. □

b) If $f(A) \subseteq f(B)$ then $A \subseteq B$.
This assertion is false. As a counterexample, consider, for $f : \mathbb{R} \to \mathbb{R}$, with $f(x) = x^2$ and with $A = [-1, 1]$ and $B = [0, 2]$, we see

$$f(A) = [0, 1] \subseteq [0, 4] = f(B) \text{ yet } A \not\subseteq B$$