MATH 225 - June 5, 2008
Exam III - 50 minutes - 50 Points

NAME:

In order to receive full credit, SHOW ALL YOUR WORK. Full credit will be given only if all reasoning and work is provided. When applicable, please enclose your final answers in boxes.

1. (10 Points) Solve the following second-order ordinary differential equations.

   (a) $y'' + y = 0$ $\Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i \Rightarrow y_h(t) = k_1\cos(t) + k_2\sin(t)$

   (b) $y'' - y = 0 \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1 \Rightarrow y_h(t) = k_1 e^{-t} + k_2 e^{t}$

   (c) $y'' = 0 \Rightarrow \lambda^2 = 0 \Rightarrow y_h(t) = k_1 e^{0t} + k_2 t e^{+0t} = k_1 + k_2 t$

2. (10 Points) Given that $y'' + 4y' + 4y = f(t)$.

   (a) Find the homogeneous solution to the ODE.

   $\lambda^2 + 4\lambda + 4 = (\lambda+2)^2 = 0 \Rightarrow \lambda = -2$

   $y_h(t) = k_1 e^{-2t} + k_2 t e^{-2t}$

   (b) Write down the form of the particular solution supposing that $f(t)$ is given by:

   i. $f(t) = 2e^{-t}$ $\quad y_p(t) = Ae^{-t}$

   ii. $f(t) = 3$ $\quad y_p(t) = A$

   iii. $f(t) = 5e^{-2t}$ $\quad y_p(t) = At^2e^{-2t}$

   iv. $f(t) = 3\cos(3t)$ $\quad y_p(t) = Ae^{3it}$ (Real part)

   DO NOT SOLVE FOR THE UNKNOWN CONSTANTS. IF USING IMAGINARY EXPONENTIALS BE SURE TO INCLUDE WHETHER THE REAL OR IMAGINARY PART SHOULD BE KEPT.

3. (10 Points) Solve the following initial value problem.

$$2y'' - 8y = 16 - 18e^{-t}, \ y(0) = 1, \ y'(0) = -3 \tag{1}$$

$y_h(t) = k_1 e^{-2t} + k_2 e^{2t}$

$y_p(t) = A + Be^{-t}$

$y_p''(t) = Be^{-t}$ $\Rightarrow 2y'' - 8y = 2Be^{-t} - 8A - 8Be^{-t} = 16 - 18e^{-t}$

$\Rightarrow A = -2$

$B = 3$

$y(t) = k_1 e^{-2t} + k_2 e^{2t} - 2 + 3e^{-t}$

$y(0) = 1 = k_1 + k_2 - 2 + 3 \Rightarrow k_1 + k_2 = 0$

$y'(0) = -2k_1 + 2k_2 - 3 = -3$ $\Rightarrow k_1 = k_2 = 0 \Rightarrow y(t) = 3e^{-t} - 2$

$\Rightarrow -2k_1 + 2k_2 = 0 \Rightarrow k_1 = k_2$

4. (10 Points) Given,

$$y' + 2y = 0. \tag{2}$$

(a) Assume a power-series solution to the ODE and find the corresponding recurrence relation for the power-series coefficients. Assume $y(t) = \sum_{n=0}^{\infty} a_n t^n$

$\Rightarrow y' + 2y = \sum_{n=0}^{\infty} a_n \cdot n \cdot t^{n-1} + 2\sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} (a_{n+1}(n+1) + 2a_n) t^n = 0$

$\Rightarrow a_{n+1} = \frac{-2a_n}{n+1}, \ n = 0, 1, \dots$

(b) Solve the recurrence relation for these coefficients and using a known Taylor series find a transcendental expression for your solution.

$n=0$ $\quad n=1$ $\quad n=2$

$a_1 = \frac{-2a_0}{1}, \quad a_2 = \frac{-2a_1}{2} = \frac{+4a_0}{2!} \quad a_3 = \frac{-2(a_2)}{3} = -\frac{8a_0}{3!}$

$\Rightarrow a_k = \frac{(-2)^k a_0}{k!} \Rightarrow y(t) = \sum_{n=0}^{\infty} \frac{(-2)^n t^n}{n!} a_0 = a_0 e^{-2t}$

(c) Check your result.

$\Rightarrow y' = -2a_0 e^{-2t} \Rightarrow y' + 2y = -2a_0 e^{-2t} + 2a_0 e^{-2t} = 0$

5. Given the following forced simple harmonic oscillator. $y_h(t) = k_1\cos(2t)+k_2\sin(2t)$

$$2\frac{d^2y}{dt^2} + 8y = 6\cos(\omega t), \quad y(0) = 0, \quad y'(0) = 0. \tag{3}$$

(a) Set $\omega = 1$ and find the solution to the initial value problem.

$y_p(t) = Ae^{it} \Rightarrow y_p''(t) = -Ae^{it}$

$\Rightarrow 2y''+8y = -2Ae^{it} + 8Ae^{it} = 6Ae^{it} = 6e^{it}$

$A = 1$

$y_p(t) = e^{it}$.

taking the Real part gives

$y_p(t) = \cos(t) \Rightarrow y(t) = k_1\cos(2t)+k_2\sin(2t)+\cos(t)$, $y(0) = k_1+1 = 0$

$k_1 = -1$

$\Rightarrow y(t) = -\cos(2t)+\cos(t)$ $\quad y'(0) = 0 = 2k_2 \Rightarrow k_2 = 0$

(b) Set $\omega = 2$ and find the solution to the initial value problem.

$y_p(t) = Ate^{2it}$

$y_p'(t) = Ae^{2it} + 2iAte^{2it}$

$y_p''(t) = 4iAe^{2it} + -4Ate^{2it}$

$\Rightarrow 2y''+8y = 8iAe^{2it} - 8Ate^{2it} + 8Ate^{2it} = 6e^{2it}$

$\Rightarrow A = -\frac{3}{4}i$

$y_p(t) = -\frac{3}{4}ite^{2it}$ taking the Real part gives.

$y_p(t) = -\frac{3}{4}t\cos(2t)$

$y(0) = 0 = k_1$

$y'(0) = 2k_2 - \frac{3}{4} = 0 \Rightarrow k_2 = \frac{3}{8} \Rightarrow y(t) = \frac{3}{8}\sin(2t) - \frac{3}{4}t\cos(2t)$

(c) Describe the qualitative differences between these solutions.

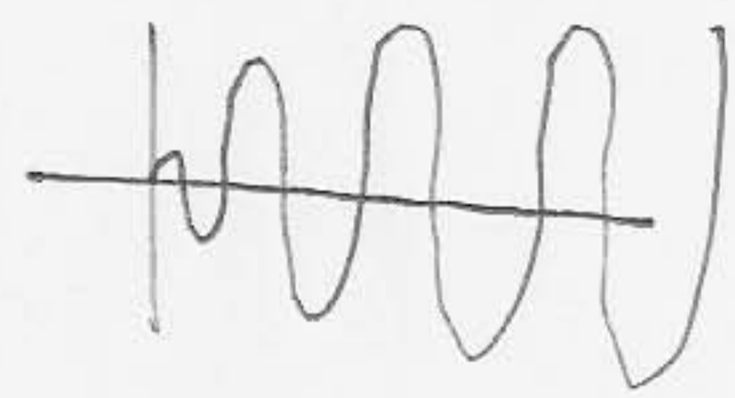
a) $\lim_{t\to\infty} y(t) < 0$

Bounded Osillations

b) $\lim_{t\to\infty} y(t) \to \infty$

Resonance

Unbounded Oscillations



6. (10 Points - Extra Credit) Given,

$$y'' + y = 0, \quad y(0) = -1, \; y'(0) = 1. \tag{4}$$

(a) Convert the second-order ODE into a system of first order ODE's.

See Quiz #3.

(b) Using eigenvalues and eigenvectors, solve the corresponding initial value problem. Express you solution in real form.