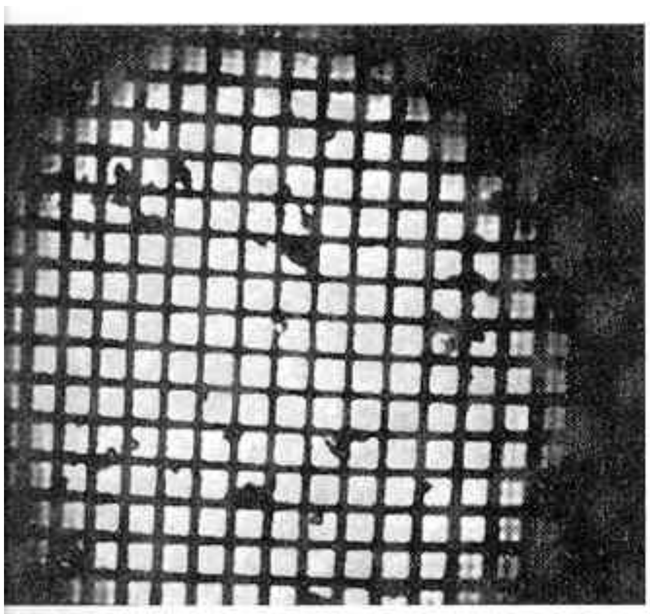
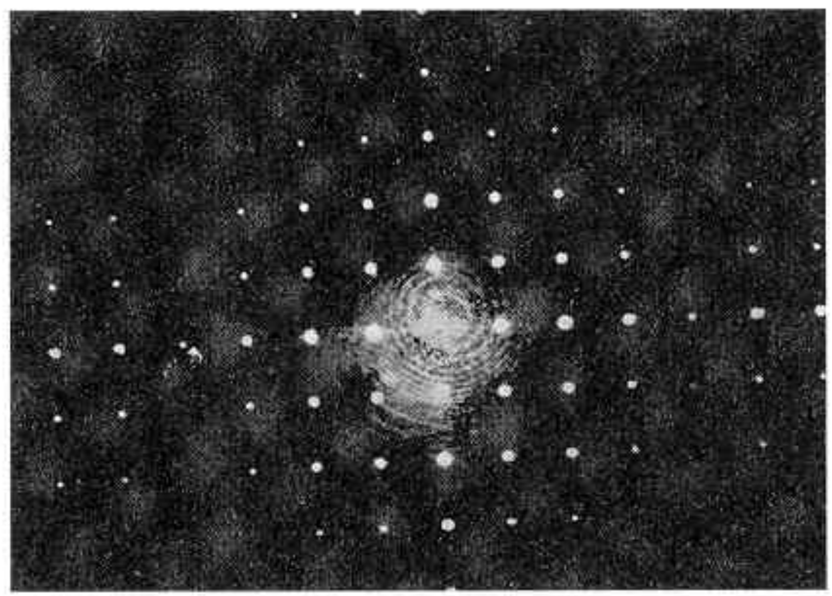
PHGN570: Physical and Fourier Optics

Today:
Transform pairs
Transform theorems





Fourier transforms: t-ω domain

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{+i\omega t}\, dt = FT\left\{ f(t) \right\}$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega t}\, dt = FT^{-1}\left\{ F(\omega) \right\}$$

- In EM, our signals are complex fields
- 1/2π factor is lumped into inverse transform
- ω is our frequency variable, not ν. This affects the normalization constants.
- Note signs of exponents: this is tied to our exp(-i ω t) convention
- Techniques
  - Analytic: apply transform IDs and theorems to decompose a transform into its parts
  - Analytic in Mathematica: can do some FTs but not always expressed in recognizable way
  - Graphical: after identifying components of a transform, sketch the anticipated result
  - Numerical: FFT for calculating complicated or realistic cases for modeling/data analysis

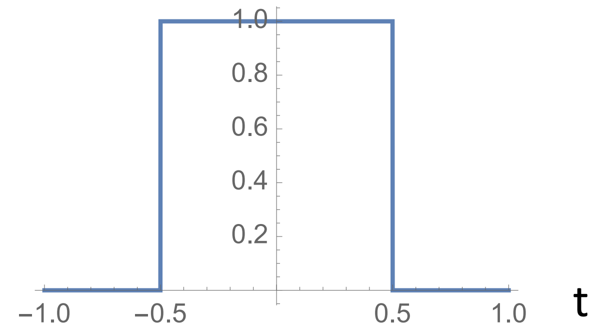
Transform pair: rect-sinc

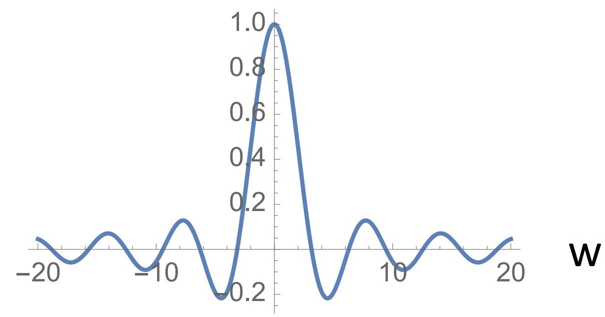
- Rect(t/t_0) $\quad rect\left(\frac{t}{t_0}\right) = 1 \text{ for } |t| < \frac{t_0}{2}$

$$F(\omega) = \int_{-\infty}^{\infty} rect(t/t_0) e^{+i\omega t}\, dt = \int_{-t_0/2}^{t_0/2} e^{+i\omega t}\, dt$$

$$= \frac{1}{i\omega}\left(e^{+i\omega t_0/2} - e^{-i\omega t_0/2}\right)$$

$$= t_0 \frac{\sin(\omega t_0/2)}{\omega t_0/2} = t_0 \operatorname{sinc}(\omega t_0/2)$$





FT of a Gaussian pulse

- Starting integral: $\int_{-\infty}^{\infty} e^{-z^2}\, dz = \sqrt{\pi}$
  - True even if z is complex

$$f(t) = e^{-t^2/t_0{}^2} \qquad FT\left\{f(t)\right\} = F(\omega) = \int_{-\infty}^{\infty} e^{-t^2/t_0{}^2} e^{+i\omega t}\, dt$$

- Complete the square in the exponent…

FT of a Gaussian is a Gaussian

- Starting integral: $\int_{-\infty}^{\infty} e^{-z^2}\, dz = \sqrt{\pi}$
  - True even if z is complex

$$f(t) = e^{-t^2/t_0{}^2} \qquad FT\left\{f(t)\right\} = F(\omega) = \int_{-\infty}^{\infty} e^{-t^2/t_0{}^2} e^{+i\omega t}\, dt$$

- Complete the square in the exponent

$$-\frac{t^2}{t_0^2} + i\omega t = -\frac{1}{t_0^2}\left(t^2 - i\,\omega\, t\, t_0^2\right) = -\frac{1}{t_0^2}\left(\left(t - \frac{i}{2}\,\omega\, t_0^2\right)^2 + \frac{1}{4}\,\omega^2\, t_0^4\right)$$

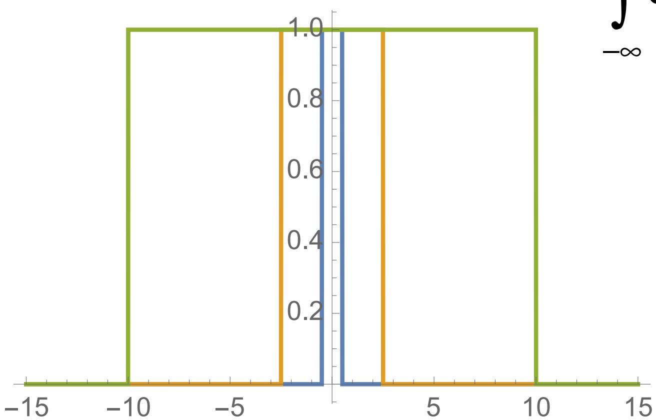
$$= -\frac{1}{t_0^2}\left(t - \frac{i}{2}\,\omega\, t_0^2\right)^2 - \frac{1}{4}\,\omega^2\, t_0^2$$

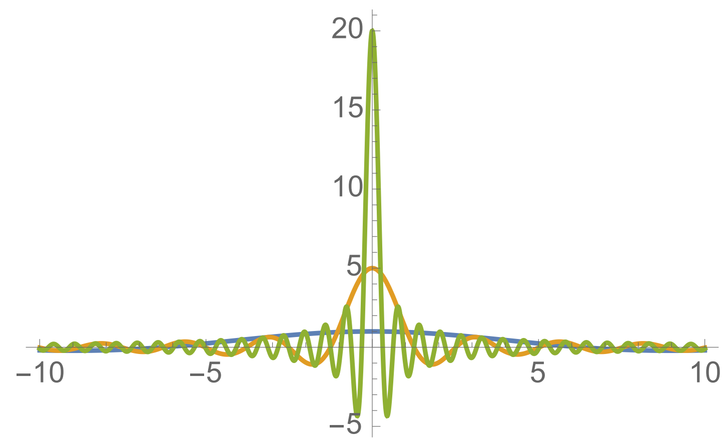
  - Change variables: $z = \frac{1}{t_0}\left(t - \frac{i}{2}\,\omega\, t_0^2\right)$

$$F(\omega) = \int_{-\infty}^{\infty} e^{-t^2/t_0^2} e^{+i\omega t}\, dt = t_0 e^{-\frac{1}{4}\omega^2\, t_0^2} \int_{-\infty}^{\infty} e^{-z^2}\, dz = \sqrt{\pi}\; t_0\; e^{-\frac{1}{4}\omega^2\, t_0^2}$$

Transform pairs: Dirac delta

- Dirac delta
  - Limit: $\delta(\omega) = \lim_{t_0 \to \infty} FT\left\{\text{rect}\left(t/t_0\right)\right\} = \lim_{t_0 \to \infty}\left[t_0 \,\text{sinc}\left(\omega t_0 / 2\right)\right]$
  - At ω=0, limit is ∞
  - ω≠0, limit is 0 in sense that integral over rapid osc sin( ) is 0
  - Normalization: $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$





$FT\{1\} = 2\pi\delta(\omega)$ $FT^{-1}\{1\} = \delta(t)$

Time-bandwidth product

- "uncertainty principle" comes from FT relations

$$FT\left(e^{-t^2/t_0^2}\right) \rightarrow t_0\ e^{-\frac{1}{4}\omega^2\ t_0^2}$$

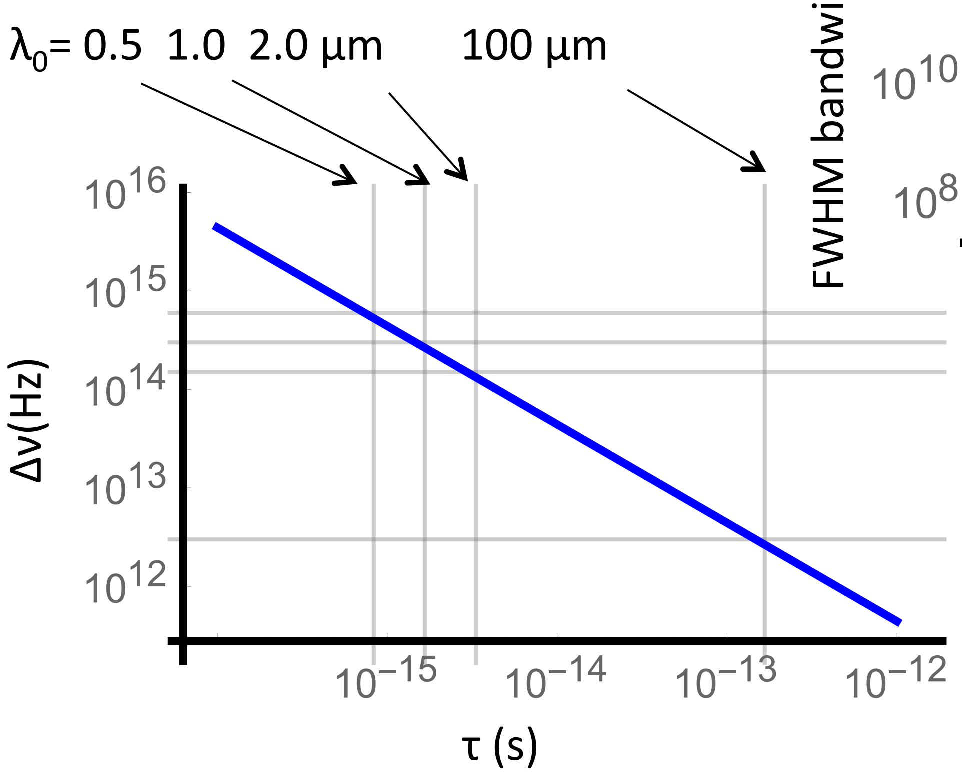
  - Pulse duration: t_0
  - Spectral width (bandwidth): $\delta\omega = 2/t_0$
  - Time-bandwidth product: $t_0\delta\omega = 2$
- This relation depends on how widths are defined
  - Here we've been using 1/e half width in the field
  - For FWHM in intensity: $E(t) = E_0 e^{-2\ln 2 t^2/\tau^2} \rightarrow I(t) \propto e^{-4\ln 2 t^2/\tau^2}$

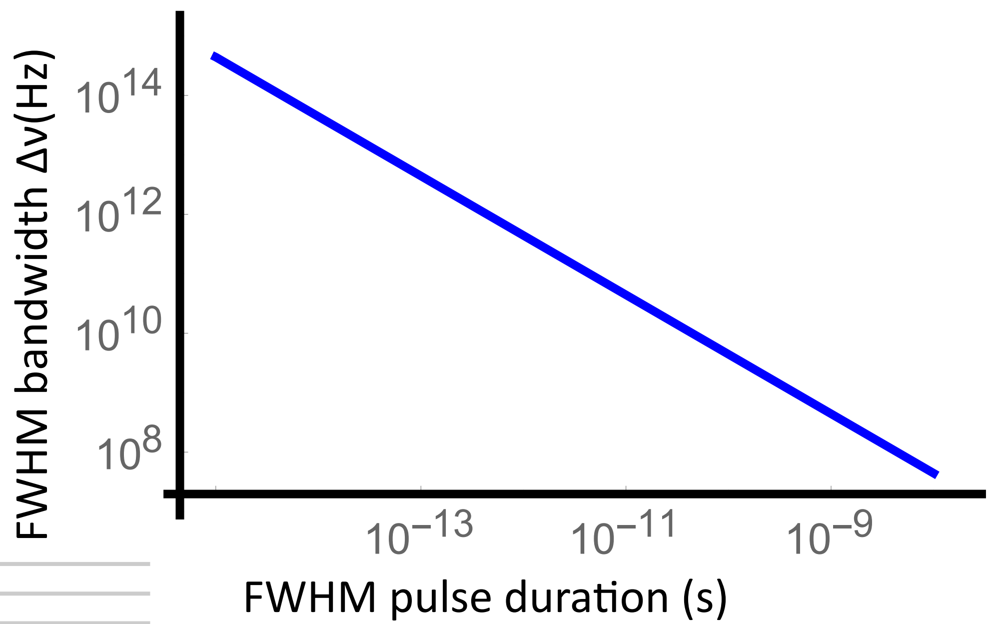
$$\tau = t_0\sqrt{2\ln 2} \qquad \Delta\omega = \delta\omega\sqrt{2\ln 2}$$

$$\boxed{t_0\delta\omega = 2 = \frac{\tau\Delta\omega}{2\ln 2} \rightarrow \tau\Delta\omega = 4\ln 2 \approx 2.77} \qquad \boxed{\tau\ \Delta\nu = \frac{4\ln 2}{2\pi} \approx 0.44}$$

Bandwidth for transform-limited pulses

- The bandwidth in frequency space is independent of the central frequency





The shortest pulse is limited to ½ of an optical cycle

Symmetry properties of FT

Symmetry Properties of Fourier Transforms

$f(x)$	$F(\xi)$
Complex, no symmetry	Complex, no symmetry
Hermitian	Real, no symmetry
Antihermitian	Imaginary, no symmetry
Complex, even	Complex, even
Complex, odd	Complex, odd
Real, no symmetry	Hermitian
Real, even	Real, even
Real, odd	Imaginary, odd
Imaginary, no symmetry	Antihermitian
Imaginary, even	Imaginary, even
Imaginary, odd	Real, odd

FT theorems

Properties of Fourier Transforms

A_1 and A_2 arbitrary constants — x_0 and ξ_0 real constants

b and d real nonzero constants — k a positive integer

$g(x)=\int_{-\infty}^{\infty} G(\beta)e^{j2\pi\beta x}d\beta$	$G(\xi)=\int_{-\infty}^{\infty} g(\alpha)e^{-j2\pi\alpha\xi}d\alpha$	
$f(\pm x)$	$F(\pm\xi)$	
$f^*(\pm x)$	$F^*(\mp\xi)$	
$F(\pm x)$	$f(\mp\xi)$	
$F^*(\pm x)$	$f^*(\pm\xi)$	
$f\left(\frac{x}{b}\right)$	$\lvert b\rvert F(b\xi)$	**scaling**
$\lvert d\rvert f(dx)$	$F\left(\frac{\xi}{d}\right)$	
$f(x\pm x_0)$	$e^{\pm j2\pi x_0\xi}F(\xi)$	**shift**
$e^{\pm j2\pi\xi_0 x}f(x)$	$F(\xi\mp\xi_0)$	

Shift theorem

- Shift in time leads to a phase shift in frequency

$$FT\left\{f\left(t-t_0\right)\right\}=\int f\left(t-t_0\right)e^{i\omega t}\,dt$$

  - Change variables: $t'=t-t_0$

$$dt=dt'$$

$$FT\left\{f\left(t-t_0\right)\right\}=\int f\left(t'\right)e^{i\omega\left(t'+t_0\right)}\,dt=F\left(\omega\right)e^{i\omega t_0}$$

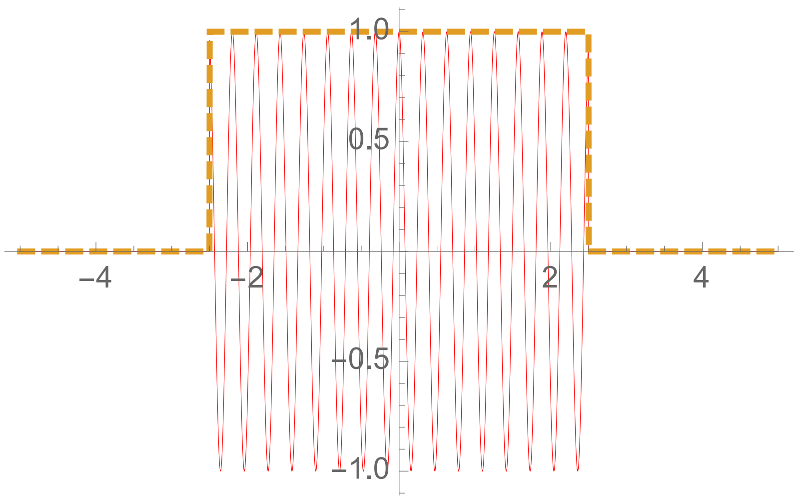
- Similarly,

$$FT^{-1}\left\{F\left(\omega-\omega_0\right)\right\}=f\left(t\right)e^{-i\omega_0 t}$$

Shift theorem: square pulse

- Gated CW laser beam

$$FT\left\{rect\left(t/t_b\right)e^{-i\omega_0 t}\right\}$$

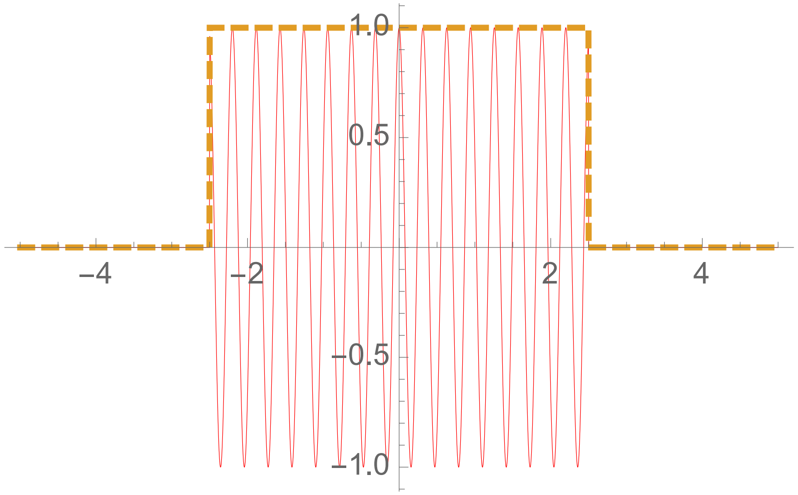


- Get expression for result, make sketch

Shift theorem: square pulse

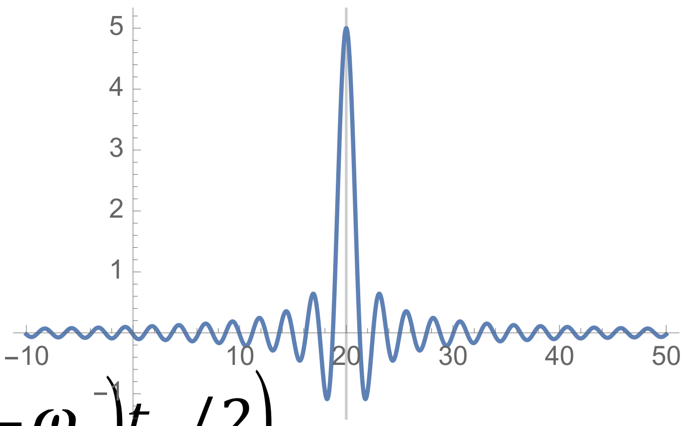
- Gated CW laser beam

$$FT\left\{rect\left(t/t_b\right)e^{-i\omega_0 t}\right\}$$



$$FT^{-1}\left\{F\left(\omega-\omega_0\right)\right\}=f\left(t\right)e^{-i\omega_0 t}$$

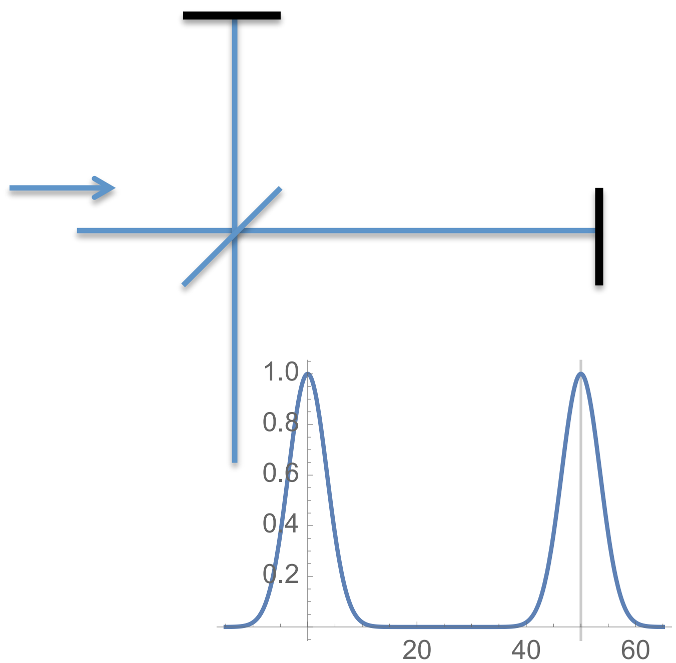
$$FT\left\{f\left(t\right)e^{-i\omega_0 t}\right\}=F\left(\omega-\omega_0\right)$$



$$FT\left\{rect\left(t/t_b\right)e^{-i\omega_0 t}\right\}\rightarrow t_b \operatorname{sinc}\left(\left(\omega-\omega_0\right)t_b/2\right)$$

Shift theorem: spectral interferogram

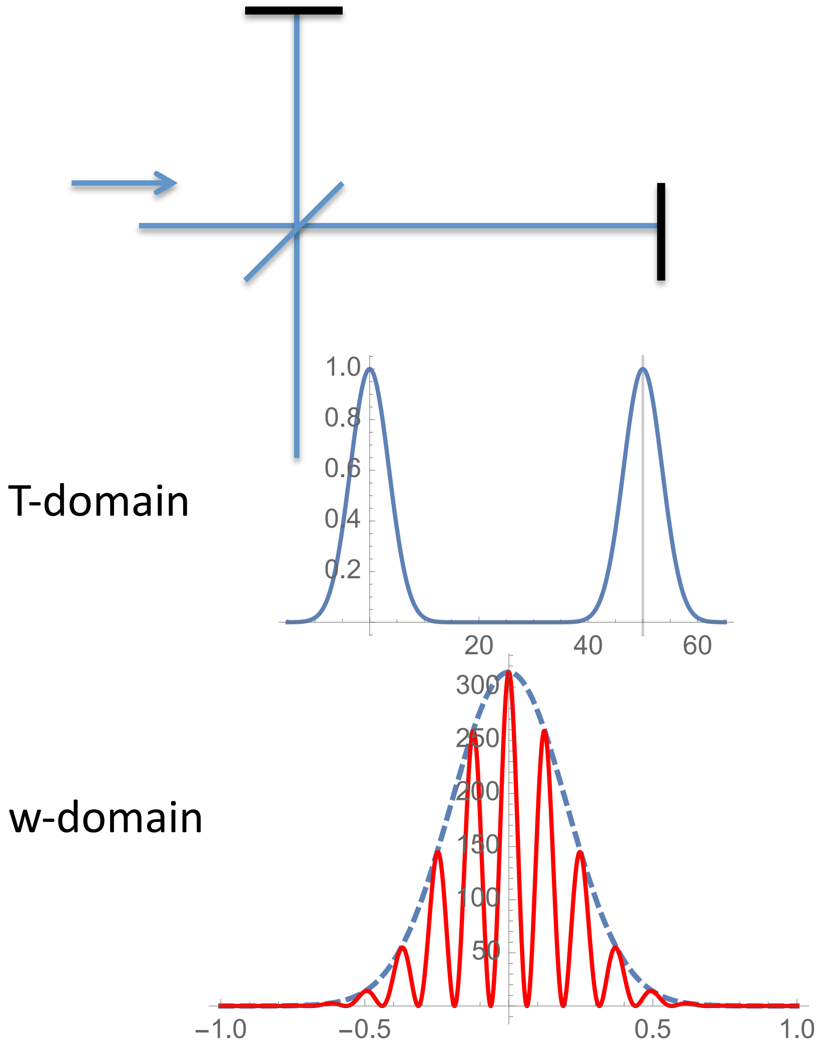
- Michelson or Mach-Zehnder interferometer output



One pulse has profile f(t)
Write $f_{tot}(t)$=two pulses
Calculate transform F(w)
Calculate intensity $\text{Abs}[F(w)]^2$

Shift theorem: spectral interferogram

- Michelson or Mach-Zehnder interferometer output



$$f_{tot}(t) = e^{-t^2/t_0{}^2} + e^{-(t-t_s)^2/t_0{}^2}$$

$$F_{tot}(\omega) = F(\omega) + F(\omega)e^{i\omega t_0}$$

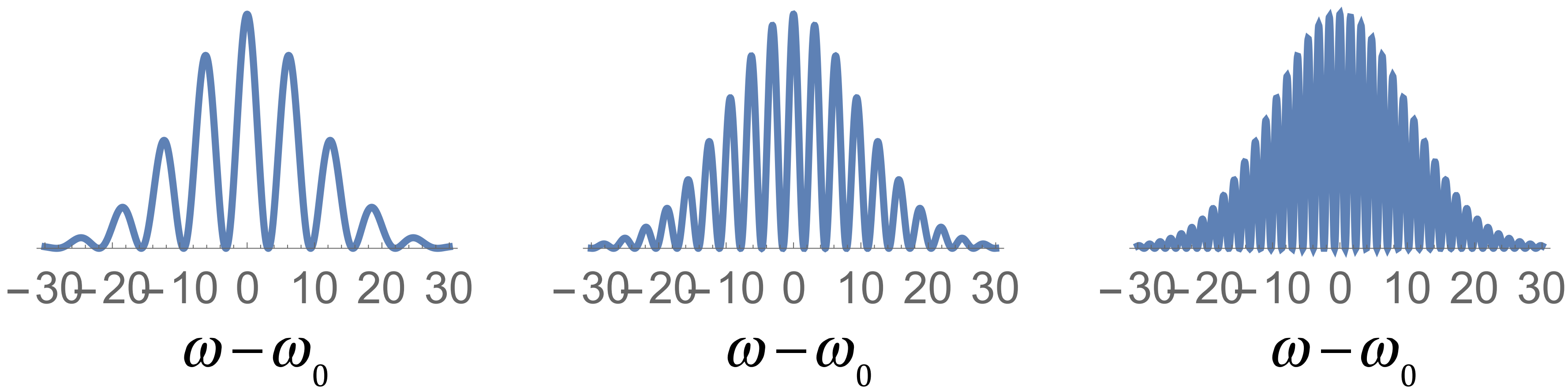
$$\left|F_{tot}(\omega)\right|^2 = \left|F(\omega)\right|^2 \left|1+e^{i\omega t_0}\right|^2$$

$$= \left|F(\omega)\right|^2 \left(2+\cos(\omega t_0)\right)$$

Two pulse spectrum

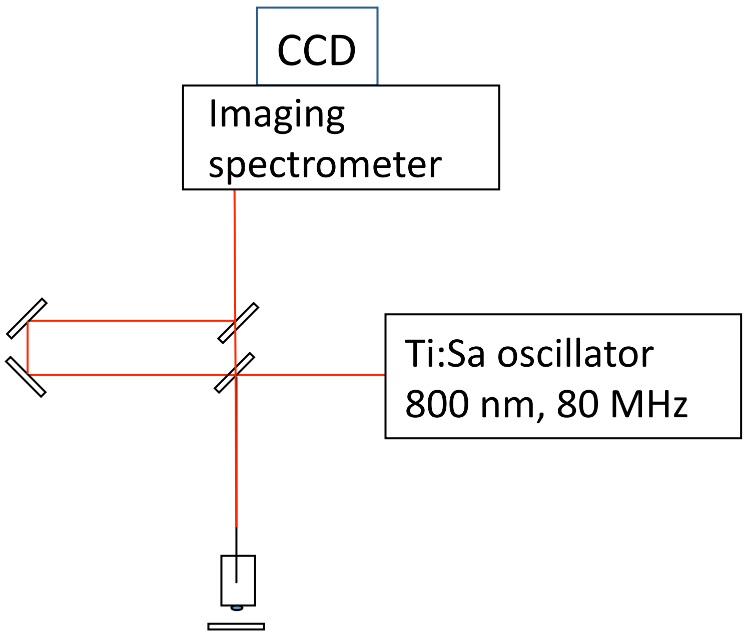
- Spectral interference of two pulses is like the double-slit interference

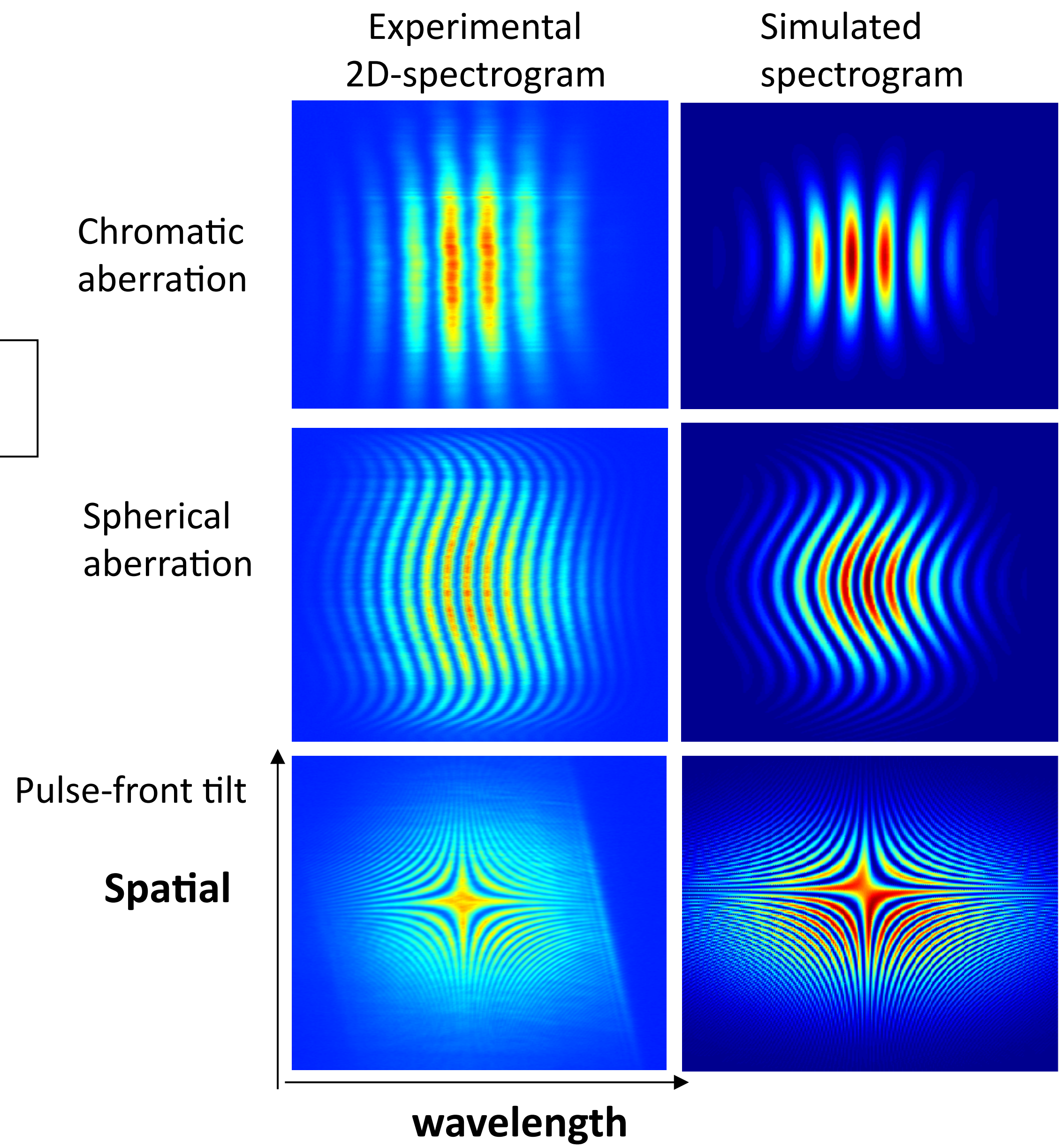
$$\left|E(\omega)+E(\omega)e^{i\omega T}\right|^2=4\left|E(\omega)\right|^2\cos^2(\omega T/2)$$



Increasing delay ➔

2D spectral interferometry





Combine spatial resolution with spectral interferometry to measure both spatial wavefront and spectral phase

Parseval's theorem

FT gives a different *representation* of the signal. Energy must be conserved.

$$\int \left|f(t)\right|^2 dt = \frac{1}{2\pi}\int \left|F(\omega)\right|^2 d\omega$$

$$= \frac{1}{2\pi}\int \left[\int f(t)e^{i\omega t}\,dt\right]\left[\int f(t')e^{i\omega t'}\,dt'\right]^* d\omega$$

Note *independent* integrals for t, t'
Apply conjugation inside integral

$$= \frac{1}{2\pi}\int \left[\int f(t)e^{i\omega t}\,dt\right]\left[\int f^*(t')e^{-i\omega t'}\,dt'\right] d\omega$$

Gather ω terms

$$= \int dt\, f(t)\int dt'\, f^*(t')\left(\frac{1}{2\pi}\int e^{i\omega(t-t')}\,d\omega\right) == \int dt\, f(t)\int dt'\, f^*(t')\delta(t'-t)$$

$$= \int dt\, f(t) f^*(t)$$

Convolution theorem

FT of the product of two functions is the convolution of the transforms

$$FT\{f(t)g(t)\} = \frac{1}{2\pi} F(\omega) \otimes G(\omega)$$

$$FT\{f(t)g(t)\} = \int f(t)g(t)e^{i\omega t}\,dt$$

$$= \int f(t)\left[\frac{1}{2\pi}\int G(\omega')e^{-i\omega' t}\,d\omega'\right]e^{i\omega t}\,dt$$

Note *independent* variables for ω, ω'

$$= \frac{1}{2\pi}\int G(\omega')d\omega' \int f(t)e^{i(\omega-\omega')t}\,dt$$

Swap order of integration: t first

$$= \frac{1}{2\pi}\int F(\omega-\omega')G(\omega')d\omega' = \frac{1}{2\pi} F(\omega) \otimes G(\omega)$$

Inverse FT of the product of two functions is the convolution of the transforms

$$FT^{-1}\{F(\omega)G(\omega)\} = f(t) \otimes g(t)$$

Graphical approach to convolution

Input functions

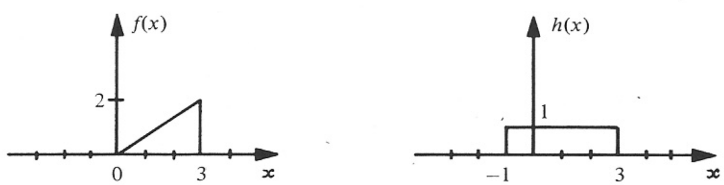


Figure 6-1 Functions used to illustrate convolution by graphical methods.

output

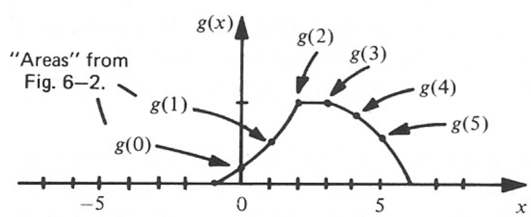


Figure 6-3 Resulting convolution of functions shown in Fig. 6-1.

Graphical view

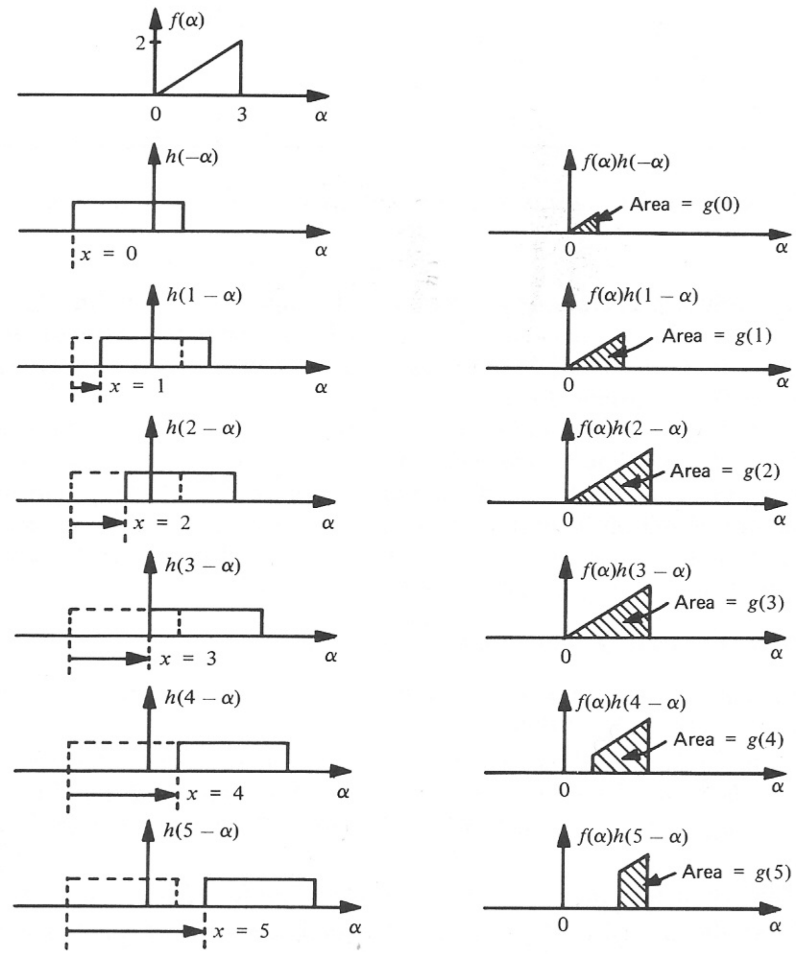


Figure 6-2 Graphical method for convolving functions of Fig. 6-1.

Smoothing effects

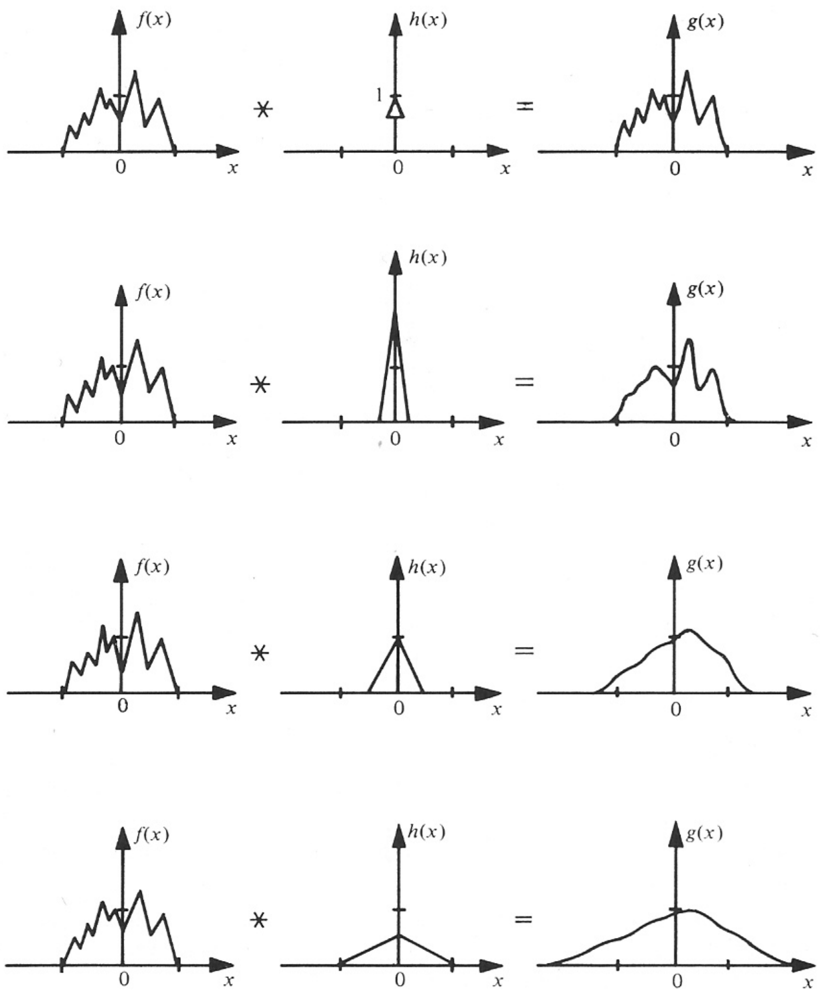


Figure 6-5 Smoothing effects of convolution.

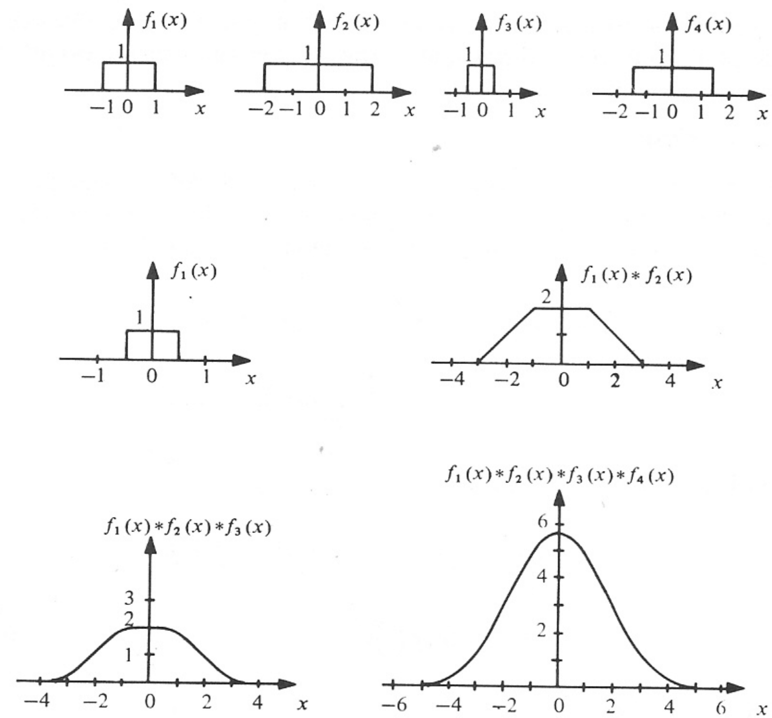


Figure 6-7 Repeated convolution of four rectangle functions.

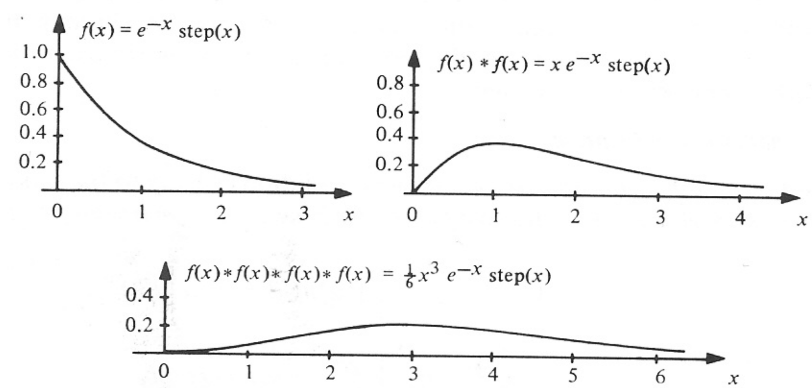


Figure 6-8 Repeated convolution of the function $\exp\{-x\}\text{step}(x)$.