Test 1 Solutions

1) a) You have a rocket in free space, whose speed is given by $v = u\ln\left(\frac{m_0}{m}\right)$. At what mass in the momentum maximum.

$$p = mv = mu\ln\left(\frac{m_0}{m}\right)$$

$$\frac{dp}{dm} = u\ln\left(\frac{m_0}{m}\right) + \cancel{m}\cancel{u}\left(\frac{m}{m_0}\right)\cdot\left(\frac{-m_0}{m^2}\right) = 0 \rightarrow \ln\left(\frac{m_0}{m}\right) = 1$$

$$\rightarrow \frac{m_0}{m} = e^1 \rightarrow \boxed{m = \frac{m_0}{e}}$$

since initially p is increasing, I know this is a max, not a min.

b) u = const of time, $m = m_0 e^{-t/t_0}$, at $t = ?$ is p max?

$$m(t_f) = \frac{m_0}{e} = m e^{-t/t_0} \rightarrow \frac{t}{t_0} = 1 \rightarrow \boxed{t = t_0}$$

2) $x(t) = x_0\sqrt{\cos(\omega t)}$ $\quad F(t) = ?$

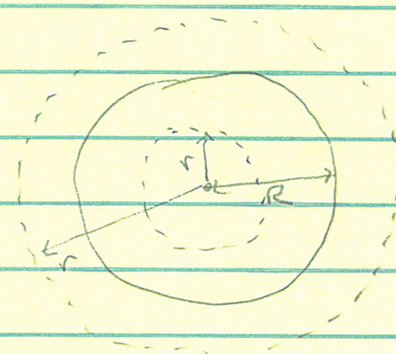
$$\dot{x} = \tfrac{1}{2}x_0(\cos(\omega t))^{-1/2}(-\sin(\omega t))\,\omega = -\tfrac{1}{2}x_0\omega\sin(\omega t)[\cos(\omega t)]^{-1/2}$$

$$\ddot{x} = -\tfrac{1}{2}x_0\omega\frac{\cos(\omega t)\,\omega}{(\cos(\omega t))^{1/2}} - \tfrac{1}{2}x_0\omega\sin(\omega t)\left[-\tfrac{1}{2}(\cos(\omega t))^{-3/2}(-\sin(\omega t))\,\omega\right]$$

$$\ddot{x} = -\tfrac{1}{2}x_0\omega^2\sqrt{\cos(\omega t)} - \tfrac{1}{4}x_0\omega^2\sin^2(\omega t)[\cos(\omega t)]^{-3/2}$$

$$\boxed{F = m\ddot{x} = -\frac{m x_0\omega^2}{4}\sqrt{\cos(\omega t)}\left(2 + \frac{\sin^2(\omega t)}{\cos^2(\omega t)}\right)}$$

3) Use Gauss' Law to calculate the field inside and outside a hollow spherical shell of mass M, radius R.



$$\oint \vec{g}\cdot d\vec{a} = -4\pi G m_{enc}$$

Inside $\oint \vec{g}\cdot d\vec{a} = 0 = g4\pi r^2 = 0 \rightarrow g = 0$

By symmetry $g \parallel d\vec{a}$ and const. $g = g\hat{r}$

$$\boxed{\vec{g} = 0 \quad r < R}$$

Outside: $g4\pi r^2 = -4\pi GM \rightarrow g = -\frac{GM}{r^2}$

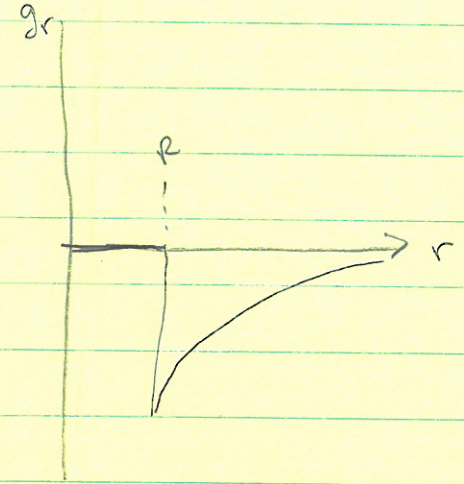
$$\boxed{\vec{g} = -\frac{GM}{r^2}\hat{r} \quad r > R}$$

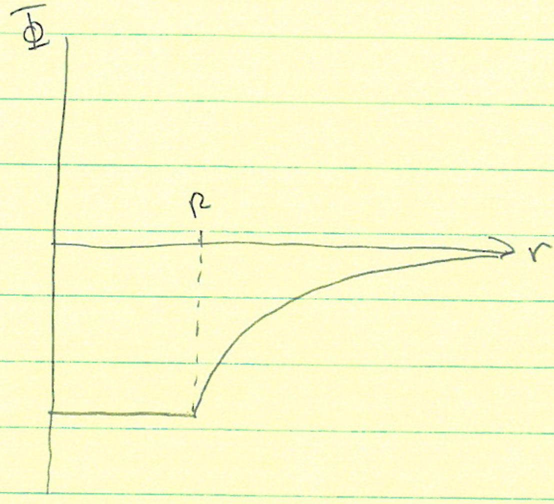
3b) Plot $g_r(r)$, $\Phi(r)$:

$$\Phi(r) = -\int_{\infty}^{r} -\frac{GM}{r^2}\,dr \quad \text{from } -\int \vec{g}\cdot d\vec{r}\,.$$

$$= -\frac{GM}{r} \quad \text{outside.}$$

Inside $\vec{g} = \emptyset$ so Φ is const.





4) a) It's the gravitational field for a point charge (less the constants on the right side of the differential equation).

b) $$x(t) = \int_{-\infty}^{t} F(t')\,G(t,t')\,dt' = \int_{0}^{t} F_0 \sin(\omega_0 t')\,A \sin(\omega_0(t-t'))\,dt'$$

$$= F_0 A \int_0^t \sin(\omega_0 t')\sin(\omega_0(t-t'))\,dt' \qquad \sin(x)\sin(y) = \tfrac{1}{2}[\cos(x-y) - \cos(x+y)]$$

$$= F_0 A \int_0^t \tfrac{1}{2}\left(\cos[\omega_0 t' - (\omega_0(t-t'))] - \cos[\omega_0 t' + \omega_0(t-t')]\right) dt'$$

$$= \frac{F_0 A}{2}\int_0^t \cos(\omega_0(2t'-t)) - \cos(\omega_0 t)\,dt'$$

$$= \frac{F_0 A}{2}\left[\frac{1}{2\omega_0}\sin(\omega_0(2t'-t)) - \cos(\omega_0 t)\,t'\right]\Big|_0^t$$

$$= \frac{F_0 A}{2}\left[\frac{1}{2\omega_0}\left(\sin(\omega_0 t) - \sin(-\omega_0 t)\right) - \cos(\omega_0 t)\,t\right]$$

$$x(t) = \boxed{\frac{F_0 A}{2}\left[\overset{\text{const}}{\frac{1}{\omega_0}}\sin(\omega_0 t) - \overset{\text{linear}}{t}\cos(\omega_0 t)\right]}$$ so amplitude grows linearly with t.

4) c) $x(t) = \int F_x(t')\, G(t,t')\, dt'$

or

$m\ddot{x} + kx = F_x(t)$ with $m\frac{d^2G}{dt^2} + kG = \delta(t-t')$

or in other words a unit impulse at $t=t'$. This means $G(t,t')$ is $x(t,t')$ given a unit impulse at $t=t'$ {means $\int F dt = 1$}.

$x=0,\ v=0$ before the impulse $m\Delta v_x = \int F_x dt$ over the impulse. so $v_x = \frac{1}{m}$ after the impulse, and $F_x = 0$ after so that means $G(t,t')$ is just the solution where $x=0$ before $t=t'$ $(t<t')$ and then $v_x = \frac{1}{m}$ directly after.

$$\boxed{G(t,t') = 0 \ ;\ t<t'}$$
$$A\sin(\omega_0 t + \delta)\ ;\ t>t'$$

$x(t=t') = \emptyset \Rightarrow A = \emptyset$ or $\omega_0 t' + \delta = \emptyset, \pi, 2\pi \ldots$ Choose $\emptyset$.

$\Rightarrow \delta = -\omega_0 t'$

$v_x(t=t') = A\omega_0 \cos(\omega_0 t' - \omega_0 t') = \frac{1}{m} = A\omega_0$

$$\Rightarrow G(t,t') = A\sin(\omega_0(t-t')) \qquad \boxed{A = \frac{1}{m\omega_0}}$$

$$\boxed{G(t,t') = \frac{1}{m\omega_0}\sin(\omega_0(t-t'))\ ;\ t>t'}$$

Units? $x(t) = \underbrace{\int \underbrace{F_x(t')}_{\left[\frac{kg\,m}{s^2}\right]}\ \underbrace{G(t,t')}_{\left[\frac{s}{kg}\right]}\ \underbrace{dt'}_{[s]}}_{[m]}$

[m] which is what it should be.

Note some of the intermediate units such as $v_x = \frac{1}{m}$ don't seem to work. This is because we put in $F_x = \delta(t-t')$ which had units of $\frac{1}{sec.}$, which isn't really a force. In the end though, the units for the answer has to work.