9/19/12

Soln to common PDE:

1.1: Show $u(x,t) = f(x-ct) + g(x+ct)$

is a soln to $u_{tt} = c^2 u_{xx}$. See Annotations for problem 3.

1.2:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\Big[e^{-4w^2 t} \underbrace{\sin(wx)}_{\text{Fourier mode}} \Big] = \sin(wx) \frac{\partial}{\partial t}\Big[e^{-4w^2 t} \Big]$$

$$= -4w^2 e^{-4w^2 t} \sin(wx)$$

Similarly,

$$\frac{\partial u}{\partial x} = w\cos(wx) e^{-4w^2 t}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = -w^2 \sin(wx) e^{-4w^2 t}$$

$$\Rightarrow u_t = \underbrace{-4w^2 e^{-4w^2 t} \sin(wx)}_{u_t} = c^2 \overbrace{\left(-w^2 \sin(wx) e^{-4w^2 t} \right)}^{u_{xx}}$$

$$\Rightarrow c^2 = 4 \Rightarrow c = \pm 2 \Rightarrow$$ u solves $u_t = c^2 u_{xx}$ if $c = 2$

1.3: PDE is $\Delta u = u_{xx} + u_{yy} + u_{zz} = 0$ 9/19/2012

Option 1: Straight up

$$u(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}} \Rightarrow u_x = \frac{x}{(x^2+y^2+z^2)^{3/2}}$$

$$\Rightarrow u_{xx} = \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3x^2}{(x^2+y^2+z^2)^{5/2}}$$

$$\Rightarrow u_{xx}+u_{yy}+u_{zz} = \frac{3}{(x^2+y^2+z^2)^{3/2}} - \underbrace{\frac{3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}}}_{3(x^2+y^2+z^2)^{-3/2}} = 0$$

Option 2: Multivar. Chain Rule

$$u(x,y,z) = u(r) = \frac{1}{r}, \quad r^2 = x^2+y^2+z^2$$

$$\Rightarrow u_x(r) = u_r r_x \Rightarrow u_{xx} = u_{rx} r_x + u_r r_{xx} =$$

$$= u_{rr}(r_x)^2 + u_r r_{xx}$$

$$\Rightarrow \Delta u = u_{rr}(r_x^2 + r_y^2 + r_z^2) + u_r(r_{xx} + r_{yy} + r_{zz})$$

Where

$$r_x = \frac{1}{2} \cdot \frac{2x}{(x^2+y^2+z^2)^{1/2}} = \frac{x}{r}.$$

$$\Rightarrow \quad r_{xx} = \frac{1}{r} - \frac{x}{r^2} \cdot r_x = \frac{1}{r} - \frac{x^2}{r^3}$$

$$\Rightarrow \quad r_{xx} + r_{yy} + r_{zz} = \frac{3}{r} - \left(\frac{x^2+y^2+z^2}{r^3} \right) = \frac{3}{r} - \frac{r^2}{r^3} = \frac{2}{r}$$

and

$$r_x^2 + r_y^2 + r_z^2 = \frac{x^2+y^2+z^2}{r^2} = 1$$

thus in the radial variable

$$0 = \Delta u = u_{rr} + \frac{2}{r} u_r$$

and

$$u(r) = \frac{1}{r} \Rightarrow u_r = -\frac{1}{r^2}, \quad u_{rr} = +\frac{2}{r^3}$$

$$\Rightarrow \quad u_{rr} + \frac{2}{r} u_r = \frac{2}{r^3} + \frac{2}{r} \cdot \left(-\frac{1}{r^2} \right) = 0 \checkmark$$

Ah, the multivar. chain rule, how I've missed you.

1.4: PDE: $\Delta u = f(x,y)$ in 2D

$u(x,y) = x^4 + y^4 \Rightarrow u_{xx} = 4 \cdot 3x^2,\ u_{yy} = 4 \cdot 3y^2$

$\Rightarrow u_{xx} + u_{yy} = 12(x^2+y^2) = f(x,y)$