Bessel's Eqn and the gamma fn:

9/1/2012

In the following we solve a general version of problem 4 from HW#1, spring 2012.

This Eqn is important in the study of linear PDE in polar settings but to do generally requires 2 tools from Math:

1) Frobenius Method (used for solving ODE with non-analytic coeff.)
2) Gamma fn (used to generalize the factorial)

The first of these has the associated theorem: [Page 180 from E.K. 10th Ed]

<u>Theorem (Frobenius)</u>: Let $b(x), c(x)$ be any $\underbrace{\text{analytic}}$ fn at $x=0$. Then the ODE

a continuous fn ~~...~~ whose derivative at all orders is also continuous.

$$(*) \quad y'' + \frac{b(x)}{x}y' + \frac{c(x)}{x^2}y = 0$$

has at least 1 Soln of the form

$$(**) \quad y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$$

where $r \in \mathbb{R}$ is chosen so that $a_0 \neq 0$.

Proof of this is technical and a matter of analysis.

while the second will be discussed when needed in the derivation of Bessel's fn of the first Kind.

Note:

- Since this is more general that problem 4, I will highlight along the way the differences.

We begin with the 2^{nd} order, homogeneous ODE w/ variable coeff. known as Bessel's Eqn of order ν [nu] (Page 187 E.K 10th ed.

$$x^2y'' + xy' + (x^2-\nu^2)y = 0 \quad , \quad \nu \in \mathbb{R}$$

which is of Frobenius form (*) at $x=0$ where $b(x)=1$, $c(x)=x^2-\nu^2$ are the analytic fn of the hypothesis. Thus, we search for Soln of the form (**).

$$x^2y'' + xy' + (x^2-\nu^2)y =$$

Note: (Purple) These multiplications all lead to x^{n+r} terms.

$$= \underline{\underline{x^2}} \sum_{n=0}^{\infty} a_n \underbrace{(n+r)(n+r-1)}_{(n+r)(n+r)-(n+r)} x^{n+r\underline{\underline{-2}}} +$$

$$+ \underline{\underline{x}} \sum_{n=0}^{\infty} a_n \underline{(n+r)} x^{n+r\underline{\underline{-1}}} - \nu^2 \sum_{n=0}^{\infty} a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} =$$

$$= \sum_{n=0}^{\infty} a_n \left[\underline{\underline{(n+r)^2}} \underline{-(n+r)} + \underline{(n+r)} - \nu^2 \right] \underline{\underline{x^{n+r}}} + \underbrace{\sum_{n=0}^{\infty} a_n x^{n+r+2}}_{\sum_{n=2}^{\infty} a_{n-2} x^{n+r}} =$$

$$= a_0(r^2-\nu^2)x^r + a_1[(r+1)^2-\nu^2]x^{1+r} +$$

$$+ \sum_{n=2}^{\infty} \left(a_n[(n+r)^2-\nu^2] + a_{n-2} \right) x^{n+r} = 0$$

Again, since power fn are linearly independent the only way to satisfy this Relation is to make all coeff. vanish (i.e. $\to 0$). However r must be chosen so $a_0 \neq 0$ thus,

In Prob. 4 $\nu = 0$ $\Rightarrow r = 0 \Rightarrow$ Power series guess will ~~not~~ work!

$$\left\{ r^2 - \nu^2 = 0 \Rightarrow r = \pm|\nu| \right.$$

If $r = -\nu$ then $y(x) \to \infty$ as $x \to 0$.

for physical reasons $\boxed{\text{Choose } r = |\nu|}$

while this is important it will not be useful to us.

If $r = \nu$ then we see

$$a_1\left[(\nu+1)^2 - \nu^2\right] = a_1\left[\nu^2 + 2|\nu| + 1 - \nu^2\right] =$$

$$= a_1\left[\underbrace{2|\nu| + 1}_{\neq 0}\right] = 0 \Rightarrow \boxed{a_1 = 0}$$ *Same in prob 4.

Lastly, we have the recursion formula

* $a_n = \dfrac{-a_{n-2}}{n^2}$

or

$a_{n+2} = \dfrac{-a_n}{(n+2)^2}$

$$a_n = \frac{-a_{n-2}}{(n+r)^2 - \nu^2} = \frac{-a_{n-2}}{(n+|\nu|)^2 - \nu^2} =$$

$$= \frac{-a_{n-2}}{n^2 + \nu^2 + 2n\nu - \nu^2} = \frac{-a_{n-2}}{n(n+2|\nu|)}, \quad n = 2, 3, 4, \ldots$$

We are now ready to calculate coeff. For ease of notation, lets assume $\nu \in \mathbb{R}^{+}$, then

$a_n = \dfrac{-a_{n-2}}{n(n+2\nu)}$ and since $a_1 = 0 \Rightarrow a_3 = 0 \Rightarrow a_{2k+1} = 0$

If $n = 2$, $n = 4$

$$a_2 = \frac{-a_0}{2(2+2\nu)} \qquad a_4 = \frac{-a_2}{4(4+2\nu)} = \frac{+a_0}{4\cdot 2\cdot(2+2\nu)(4+2\nu)}$$

$n=6$,

$$a_6 = \frac{-a_4}{6(6+2\nu)} = \frac{-a_0}{\underbrace{6}_{2\cdot 3}\cdot\underbrace{4}_{2\cdot 2}\cdot\underbrace{2}_{2\cdot 1}\cdot\underbrace{(6+2\nu)(4+2\nu)(2+2\nu)}_{2\cdot 2\cdot 2\,(3+\nu)(2+\nu)(1+\nu)}} =$$

$$= \frac{-a_0}{3\cdot 2\cdot 1\cdot 2^6\,(\nu+3)(\nu+2)(\nu+1)}$$

This generalizes to

$$a_{2k} = \frac{(-1)^k a_0}{k!\; 2^{2k}\,\underbrace{(\nu+k)(\nu+k-1)(\nu+k-2)\cdots(\nu+1)}_{(*)}}$$

(*) If $\nu = 0$ then this is $k!$

* For prob. 4 this becomes

$$a_{2k} = \frac{(-1)^k a_0}{(k!)^2\, 2^{2k}}$$

Q: If $\nu \neq 0$ or if ν is not a postive whole number then what is (*)?

Defn: (Gamma fn) Let $\nu \in \mathbb{R}^+$ then define (Pg 196 of E.K. 10th ed)

$$\Gamma(\nu+1) = \int_0^\infty e^{-t} t^\nu \, dt.$$

Properties:

- $\Gamma(1) = \int_0^\infty e^{-t} \, dt = 1$

$u = t^\nu \qquad dv = e^{-t} dt$

$du = \nu t^{\nu-1} dt \qquad v = -e^{-t}$

- $$\Gamma(\nu+1) = \int_0^\infty e^{-t} t^\nu \, dt = \underbrace{-e^{-t} t^\nu \Big|_0^\infty}_{0} - \int_0^\infty -\nu e^{-t} t^{\nu-1} \, dt =$$

by L'Hôpital's rule

$$= \int_0^\infty \nu e^{-t} t^{\nu-1} \, dt = \nu \Gamma(\nu)$$

$\Rightarrow$ Point! $\Gamma(\nu+1) = \nu\Gamma(\nu)$ } Recursion structure of factorials!

Thus if ν is a positive integer, we get

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) =$$

$$= n(n-1)(n-2)\Gamma(n-2) \cdots n(n-1)(n-2)\cdots 1 \, \Gamma(1) =$$

$= n!$ * This fn Γ generalizes the factorial fn to non-integer inputs!

Now we ~~ke~~ can say

$$(\nu+k)(\nu+k-1)(\nu+k-2)\cdots(\nu+1)\Gamma(\nu+1) =$$

$$= \Gamma(\nu+k+1)$$

$\Rightarrow$

$$(\nu+k)(\nu+k-1)(\nu+k-2)\cdots(\nu+1) = \frac{\Gamma(\nu+k+1)}{\Gamma(\nu+1)}$$

and

$$a_{2k} = \frac{(-1)^k \, a_0 \, \Gamma(\nu+1)}{k! \, 2^{2k} \, \Gamma(\nu+k+1)}$$

Which defines Bessel's $f\underline{n}$ of the first Kind of order ν.

$*$ $\nu=0$ then $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$

Convention

$$J_\nu(x) = y(x) = x^\nu \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n+\nu} \, n! \, \Gamma(\nu+n+1)}$$

Pg 191 of E.K. 10th

where $a_0 \in \mathbb{R}$ is $\underbrace{\text{chosen to be}}_{}$ $a_0 = \frac{1}{2^\nu \Gamma(\nu+1)}$

Convention. Remember a_0 could be anything and any multiple of y is again a $sol\underline{n}$ to ODE.