Reading: G 10.1 (10.2)

Monday: G 10.3

Potentials

In electrostatics, you used "voltage" (V, or ϕ in most advanced books), sometimes called the scalar potential. Also, there was $\vec{A}$, the vector potential.

$$W = \int \vec{F}\cdot d\vec{\ell} = \int \underbrace{q\vec{E}}_{\text{force just from } \vec{E}\text{-field.}} \cdot d\vec{\ell} = q\int \vec{E}\cdot d\vec{\ell} = -q\Delta V$$

Relationship between V and energy.

Let's expand:

Why did we say $\vec{E} = -\vec{\nabla} V$?

$\vec{\nabla}\times\vec{E} = \phi$ { in e-statics ✓, in e-dynamics ✗ }

$\vec{\nabla}\times(\vec{\nabla} f) = \phi$ for any f.

$\vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}$

$\vec{B} = \vec{\nabla}\times\vec{A}$

Why? $\vec{\nabla}\cdot\vec{B} = \phi$ {✓}

and $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) = \phi$

any vector field

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$\Rightarrow$ we can still $\vec{B} = \vec{\nabla}\times\vec{A}$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A})$$

$$= \vec{\nabla}\times\left(-\frac{\partial\vec{A}}{\partial t}\right)$$

$$\Rightarrow \vec{\nabla}\times\underbrace{\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right)} = 0$$

You can write this as $\vec{\nabla}V$ or $-\vec{\nabla}V$

$$\vec{E} + \frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}V$$

$\vec{E}$ can be written as

$$\boxed{\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t} \ ; \quad \vec{B} = \vec{\nabla}\times\vec{A}}$$

Plug these definitions into all of Maxwell's eqns.

(i) $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$ (ii) $\vec{\nabla}\cdot\vec{B} = 0$

(iii) $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ (iv) $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$

(i) $\vec{\nabla}\cdot\left(-\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}\right) = \rho/\epsilon_0 = \boxed{-\nabla^2 V - \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = \rho/\epsilon_0}$

(ii) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$, trivial

(iii) $\vec{\nabla}\times\left(\cancel{-\vec{\nabla}V}^{\,0} - \frac{\partial\vec{A}}{\partial t}\right) = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A})$, trivial

(iv) $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial}{\partial t}\left(-\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}\right)$

$$\boxed{\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J} - \mu_0\epsilon_0\frac{\partial}{\partial t}(\vec{\nabla}V) - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2}}$$

If you us $\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}$; $\vec{B} = \vec{\nabla}\times\vec{A}$ are equivalent to Maxwell's eqns, except you have a little more freedom... too much.

Gauge Transformations

Let's say that

$$\vec{A}' = \vec{A} + \vec{\alpha} \quad ; \quad V' = V + \beta$$

Can I get an $\vec{\alpha}, \beta$ such that $\vec{E}$ and $\vec{B}$ are unchanged?

$$\vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \vec{A} + \vec{\nabla} \times \vec{\alpha}.$$

If $\vec{\alpha} = \vec{\nabla} \lambda$ ← any scalar function $\Rightarrow \vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \vec{A}$

we need $\vec{E}$ to not change...

$$-\vec{\nabla} V' - \frac{\partial \vec{A}'}{\partial t} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \quad \{ V' = V + \beta$$

$$-\vec{\nabla}(V + \beta) - \frac{\partial}{\partial t}(\vec{A} + \vec{\nabla}\lambda) = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

$$-\vec{\nabla}\cancel{V} - \vec{\nabla}\beta - \cancel{\frac{\partial \vec{A}}{\partial t}} - \frac{\partial}{\partial t}\vec{\nabla}\lambda = -\vec{\nabla}\cancel{V} - \cancel{\frac{\partial \vec{A}}{\partial t}}$$

$$\vec{\nabla}\beta + \frac{\partial}{\partial t}\vec{\nabla}\lambda = \emptyset$$

$$\vec{\nabla}\left(\beta + \frac{\partial \lambda}{\partial t}\right) = \emptyset$$

$$\beta = -\frac{\partial \lambda}{\partial t} + f(t)$$

↑ const with respect to position.

Let's absorb $f(t) \rightarrow \frac{\partial \lambda}{\partial t}$

You can change your potentials by using the following "gauge transformation"

$$V' = V - \frac{\partial \lambda}{\partial t} \quad ; \quad \vec{A}' = \vec{A} + \vec{\nabla}\lambda$$

Some popular gauges:

Coloumb Gauge: simplify (i) from last page by setting $\vec{\nabla}\cdot\vec{A}' = \emptyset$

(i) $-\nabla^2 V = \rho/\epsilon_0$

(iv) $\vec{\nabla}(\overset{\emptyset}{\cancel{\vec{\nabla}\cdot\vec{A}}}) - \nabla^2\vec{A} = \mu_0\vec{J} - \mu_0\epsilon_0\frac{\partial}{\partial t}(\vec{\nabla}V) - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2}$

$$-\nabla^2\vec{A} + \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} + \mu_0\epsilon_0\frac{\partial}{\partial t}(\vec{\nabla}V) = \mu_0\vec{J}$$

Lorentz gauge:

Pick λ s.t. $\vec{\nabla}\cdot\vec{A} = -\epsilon_0\mu_0\frac{\partial V}{\partial t}$

Plug this into Maxwell's eqn's (i), (iv) from prev. page, see what you get.

$$\left(\nabla^2 - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_0\vec{J}$$

$$\left(\nabla^2 - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\right)V = -\frac{\rho}{\epsilon_0}$$

$$\left.\begin{aligned}\Box^2\vec{A} &= -\mu_0 J\\ \Box^2 V &= -\rho/\epsilon_0\end{aligned}\right\} \quad \underline{\partial_\nu\partial^\nu A^\mu = J^\mu}$$

General solution

(lorentz gauge)

$$V(\vec{r},t) = \int_{\substack{\text{all}\\\text{space}}} \frac{\rho(\vec{r}\,',t_r)}{\mathscr{r}}\,d\tau' \; ; \; \vec{A}(\vec{r},t) = \int_{\substack{\text{all}\\\text{space}}} \frac{\vec{J}(\vec{r}\,',t_r)}{\mathscr{r}}\,d\tau'$$

$$t_r = t - \frac{\mathscr{r}}{c} \; ; \; \mathscr{r} = |\vec{r} - \vec{r}\,'|$$

retarded time

In electrostatics,

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'$$

You might think

$$\vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon}\int \frac{\rho(\vec{r}\,',t_r)\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'$$

wrong...

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int_{\substack{\text{all}\\\text{space}}}\left[\frac{\rho(\vec{r}\,',t_r)\hat{\mathscr{r}}}{\mathscr{r}^2} + \frac{\dot{\rho}(\vec{r}\,',t_r)\hat{\mathscr{r}}}{c\mathscr{r}} - \frac{\dot{\vec{J}}(\vec{r}\,',t_r)}{c^2\mathscr{r}}\right]d\tau'$$

$$\vec{B} = \frac{\mu_0}{4\pi}\int_{\substack{\text{all}\\\text{space}}}\left[\frac{\vec{J}(\vec{r}\,',t_r)}{\mathscr{r}^2} + \frac{\dot{\vec{J}}(\vec{r}\,',t_r)}{c\mathscr{r}}\right]\times\hat{\mathscr{r}}\,d\tau'$$